Introduction to Hodge-type Structures

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"Having thus refreshed ourselves in the oasis of a proof, we now turn again into the desert of definitions."

— Th. Brocker & K. Janich, Introduction to Differential Topology [21]

"The introduction of the cipher 0 or the group concept was general nonsense too..."

— Alexander Grothendieck, correspondence with Ronald Brown

1 Introduction

This paper aims to exposit certain structures present within the complex cohomology of quasi-projective varieties over $\mathbb{C}$, and to give generalizations of Saito to families of such varieties.

An algebraic variety over $\mathbb{C}$ is the set of common zeroes of a set of (multivariate) polynomials with complex coefficients—it naturally has a geometric structure, and indeed, apart from a lower-dimensional locus of singularity, it is a smooth complex manifold. The structure of such a variety in general is hard to understand. Thus, we examine *linear* invariants of the variety, called cohomology; these invariants are *functorial*, meaning that a map between two varieties induces a map on cohomology. In many of the simpler cases under consideration here, the cohomology has a decomposition into smaller functorial invariants, called the Hodge decomposition.

These invariants reflect many of the important properties of the variety, so it is natural to want to find similar invariants in a more general setting. For example, one might want to find such invariants for families of varieties, parametrized by another space (one might example the varieties as changing over time, for instance). The goal here is to analyze and exposit some of these more general invariants.

We now make this goal more precise. Let (M, I) be an almost complex manifold, where $I : \mathbb{T}_M \to \mathbb{T}_M$ is the endomorphism of the tangent bundle inducing the almost complex structure, that is, an endomorphism satisfying $I^2 = -1$. Let h be a Hermitian metric on M (that is, a Hermitian metric on each tangent space, with respect to the complex structure induced by I, that varies smoothly along M). Then call

$$\omega = -\operatorname{Im}(h) \in \Omega^{1,1}_{M,\mathbb{C}} \cap \Omega^2_{M,\mathbb{R}}$$

the associated Kähler form to h. We say that h is a Kähler metric if I is integrable (that is, M has a complex structure) and ω is a closed 2-form, that is,

$$d\omega = 0.$$

M is a Kähler manifold if it admits a Kähler metric. [18]

Among the most important examples of compact Kähler manifolds are smooth projective varieties; that these are Kähler follows from the fact that $\mathbb{P}^n(\mathbb{C})$ is Kähler.

Let M be a compact Kähler manifold. Then the Hodge decomposition theorem states that

$$H^k(M, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}$$

canonically, where $H^{p,q} \simeq \overline{H^{q,p}}$ canonically and $H^{p,q} \simeq H^q(M, \Omega_M^{p,0})$ canonically; that is, $H^{p,q}$ is the (p,q) Dolbeault cohomology and thus the decomposition is functorial. This decomposition is thus an invariant of the complex structure, not the Kähler structure.

In the case that M is a (possibly noncompact) complex quasiprojective variety, a Hodge structure cannot necessarily be defined; instead, one resorts to using a *mixed Hodge structure*, which, loosely speaking, consists of two filtrations W and F on the cohomology of M, such that F induces a Hodge structure on each of the components of the graded module associated to W. Deligne showed that the cohomology of a complex variety admits a mixed Hodge structure in general. [2-7]

Now consider the case of a family of compact Kähler varieties parametrized by some smooth base space B; that is, a proper holomorphic submersion $f : E \to B$ such that the variety corresponding to the point $b \in B$ is given by $f^{-1}(B)$; we assume the fibers are smooth and connected. One may consider a *variation of Hodge structure* on B to be a particular locally free sheaf of $\mathbb{C}$-vector spaces whose fibers are given by $H^n(f^{-1}(b), \mathbb{C})$; viewed as a functor, this is the n-th derived functor of f_* applied to the constant sheaf $\mathbb{C}$ over E, and admits a Hodge decomposition in the obvious way—that is, it may be filtered by sub-bundles in a way analogous to the Hodge decomposition of the cohomology of a Kähler manifold.

But what if the family is not smooth—more precisely, what if we have a dense open subset U of B parametrizing a family of smooth compact Kähler manifolds, but if for points $x \in B \setminus U$ we have that $f^{-1}(x)$ is singular? Is there structure analogous to the one above?

The answer is yes—in fact, one may define a variation of mixed Hodge structure in this case, which follows in the one-dimensional case from work of Schmid [1]. Saito [14] generalized this work in his theory of Mixed Hodge Modules, which to each smooth complex variety associates an abelian category of "mixed Hodge modules." Given a map $f : X \to Y$ of smooth complex varieties, there exist functors $f_*, f^*, f_!, f^!$ between the associated derived categories, satisfying the required adjointness properties and relationships via Verdier duality. In the case above, the mixed Hodge structure on B is given by $i_*(H)$, where $i : U \to B$ is the natural inclusion and and H is the variation of Hodge structure on U. [12-15]

This goal of this thesis was initially to compute the functors $f_*, f^*, f_!, f^!$ in the case that B is one-dimensional and U is dense and open in B; as all the constructions are local, it suffices to do this for B the open unit disc in $\mathbb{C}$ and U the punctured disc. As the prerequisites for such a computation are very high we do not quite reach this goal—instead, we exposit the basic structures and prerequisites necessary to approach this work.

For an analogous but less technical approach to this problem, see Durfee [20].

2 Review of Hodge Theory

2.1 De Rham and Dolbeault Cohomology

Let $\mathrm{Man}_{\mathbb{C}}$ be the category of smooth manifolds with complex structure, with morphisms given by holomorphic maps. For M a smooth manifold with complex structure, let $\mathbb{T}_M = \mathbb{T}_M^{1,0} \oplus \mathbb{T}_M^{0,1}$ be the tangent bundle. Let $\Omega^p(M)$ be the sheaf of holomorphic p-forms on M (that is, holomorphic sections of the complex vector bundle $\bigwedge^p(\mathbb{T}_M^{1,0})^* \underset{\mathbb{R}}{\otimes} \mathbb{C}$), and for $p, q \geq 0$, let $\Omega^{p,q}(M)$ be the sheaf of complex differential forms of degree (p,q) (that is, smooth sections of the complex vector bundle $\bigwedge^p(\mathbb{T}_M^{1,0})^* \otimes \bigwedge^q(\mathbb{T}_M^{0,1})^* \underset{\mathbb{R}}{\otimes} \mathbb{C}$). Let $\Lambda^k(M)$ be the sheaf of smooth k-forms on M, that is, smooth sections of $\bigwedge^k(\mathbb{T}_M)^* \underset{\mathbb{R}}{\otimes} \mathbb{C}$. Note that as

$$\bigwedge^k(\mathbb{T}_M)^* \simeq \bigwedge^k(\mathbb{T}_M^{1,0} \oplus \mathbb{T}_M^{0,1})^* \simeq \bigoplus_{p+q=k} \bigwedge^p(\mathbb{T}_M^{1,0})^* \otimes \bigwedge^q(\mathbb{T}_M^{1,0})^*$$

we have that

$$\Lambda^k(M) \simeq \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

Let $\Lambda^\bullet(M)$ be the chain complex given by the exterior derivative d, that is,

$$\Lambda^\bullet(M): \qquad \cdots \longrightarrow \Lambda^{k-1}(M) \xrightarrow{d} \Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \longrightarrow \cdots$$

where $\Lambda^k(M) = 0$ for $k < 0$. The Poincaré Lemma [22] asserts that this sequence is exact away from degree zero and that the kernel of $d^0 : \Lambda^0(M) \to \Lambda^1(M)$ is the constant sheaf $\mathbb{C}$; as Λ^k is a fine (and thus acyclic) sheaf for all k, this implies de Rham's theorem:

Theorem 1 (de Rham's theorem)**.** *There exists a natural isomorphism*

$$H^k(M, \mathbb{C}) \simeq H^k_{DR}(M),$$

where $H^k_{DR}(M)$ is defined as $H^k(\Gamma(M, \Lambda^\bullet(M)))$.

Note that

$$d(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$$

so we may write $d = \partial + \overline{\partial}$, where $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. As $d \circ d = 0$

we have that $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$. These relations imply that the diagram $\Omega^{\bullet,\bullet}(M)$ given by

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \Omega^{p-1,q-1}(M) \arrow[r, "\overline{\partial}"] \arrow[d, "\partial"] & \Omega^{p-1,q} \arrow[r, "\overline{\partial}"] \arrow[d, "\partial"] & \Omega^{p-1,q+1} \arrow[r] \arrow[d, "\partial"] & \cdots \\
\cdots \arrow[r] & \Omega^{p,q-1}(M) \arrow[r, "\overline{\partial}"] \arrow[d, "\partial"] & \Omega^{p,q} \arrow[r, "\overline{\partial}"] \arrow[d, "\partial"] & \Omega^{p,q+1} \arrow[r] \arrow[d, "\partial"] & \cdots \\
\cdots \arrow[r] & \Omega^{p+1,q-1}(M) \arrow[r, "\overline{\partial}"] \arrow[d] & \Omega^{p+1,q} \arrow[r, "\overline{\partial}"] \arrow[d] & \Omega^{p+1,q+1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

is a bicomplex, with associated total complex

$$\mathrm{Tot}(\Omega^{\bullet,\bullet}(M)) \simeq \Lambda^{\bullet}(M). \tag{1}$$

We let $\Omega^{p,\bullet}(M)$ and $\Omega^{\bullet,q}(M)$ denote the rows and columns of this diagram, respectively.

The Dolbeault Lemma [18] states that the rows $\Omega^{p,\bullet}$ are exact as sequences of sheaves, away from degree zero, where the kernel of $\overline{\partial} : \Omega^{p,0}(M) \to \Omega^{p,1}(M)$ is $\Omega^p(M)$, the sheaf of holomorphic p-forms on M. This implies that $\Omega^{p,\bullet}(M)$ is a fine resolution of $\Omega^p(M)$ (as the notation suggests), giving Dolbeault's theorem:

Theorem 2 (Dolbeault's theorem)**.** *There exists a natural isomorphism*

$$H^q(M, \Omega^p(M)) \simeq H^{p,q}(M, \mathbb{C})$$

where $H^{p,q}(M, \mathbb{C})$ *is defined as* $H^q(\Gamma(M, \Omega^{p,\bullet}(M)))$.

More generally, given any holomorphic vector bundle E over M with sheaf of holomorphic sections $\mathcal{E}$, let $\Omega^{p,q}(E)$ be the sheaf of smooth sections of the vector bundle $E \otimes \Omega^{p,q}$. Then the sequence

$$\Omega^{0,\bullet}(E) : \cdots \to \Omega^{0,q-1}(E) \xrightarrow{\mathrm{id}\otimes\overline{\partial}} \Omega^{0,q}(E) \xrightarrow{\mathrm{id}\otimes\overline{\partial}} \Omega^{0,q+1}(E) \to \cdots$$

is a fine resolution of $\mathcal{E}$, and thus we have

Theorem 3 (Vector Bundle Dolbeault theorem)**.** *There exists a natural isomorphism*

$$H^k(\Gamma(M, \Omega^{0,\bullet}(E))) \simeq H^k(M, \mathcal{E}).$$

Taking E to be $\Omega^p(M)$ gives the original version of Dolbeault's theorem.

In view of de Rham's theorem and Dolbeault's theorem as well as identity (1), there are several natural questions to ask. Let $I^q : \mathrm{Man}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{C}}$ be the contravariant functor sending a complex manifold M to

$$\mathrm{Im}(\Gamma(d) : \Gamma(M, \Lambda^{q-1}(M)) \to \Gamma(M, \Lambda^q(M)));$$

this is a functor as the exterior derivative commutes with the pullback. Let $K^q : \mathrm{Man}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{C}}$ be the contravariant functor sending a complex manifold M to

$$\mathrm{Ker}(\Gamma(d) : \Gamma(M, \Lambda^q(M)) \to \Gamma(M, \Lambda^{q+1}(M))).$$

Similarly, let $I^{p,q} : \mathrm{Man}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{C}}$ be the functor sending M to

$$\mathrm{Im}(\Gamma(\overline{\partial}) : \Gamma(M, \Omega^{p,q-1}(M)) \to \Gamma(M, \Omega^{p,q}(M)))$$

and let $K^{p,q} : \mathrm{Man}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{C}}$ be the functor sending M to

$$\mathrm{Ker}(\Gamma(\overline{\partial}) : \Gamma(M, \Omega^{p,q}(M)) \to \Gamma(M, \Omega^{p,q+1}(M))).$$

As before, these functors are contravariant and send a morphism $f : M \to N$ to the pullback f^*.

Then we have short exact sequences of functors

$$0 \to I^q \to K^q \to H^q_{DR} \to 0 \tag{2}$$

$$0 \to I^{p,q} \to K^{p,q} \to H^{p,q}(-, \mathbb{C}) \to 0 \tag{3}$$

We may ask:

Do these short exact sequences split canonically?

In view of theorem 3, we may ask a similar question for the cohomology of any holomorphic vector bundle over M. The answer to all these questions is yes, in some generality; here we answer it for compact Hermitian manifolds. Furthermore, in view of the spectral sequence of the double complex $\Omega^{\bullet,\bullet}$ and identity (1), we may ask:

Is there a simple statement of the relationship between $H^{p,q}(M,\mathbb{C})$ and $H^k_{DR}(M)$?

This latter question has an answer in the case of Kähler manifolds.

2.2 The Hodge Theorem

2.2.1 A Model Case

We will first turn our attention to the first question—that is, to the splitting of the sequences (2) and (3). Before we do so, we will motivate the solution with an analysis of a model case with a finite-dimensional complex.

Lemma 1. *Let V be a finite dimensional inner product space and $\Delta : V \to V$ a self-adjoint operator. Then there is a natural (orthogonal) decomposition*

$$V \simeq \mathrm{Ker}(\Delta) \oplus \mathrm{Im}(\Delta).$$

Proof. By dimension-counting, it suffices to show that $\mathrm{Ker}(\Delta) \perp \mathrm{Im}(\Delta)$. Consider $v \in \mathrm{Im}(\Delta), v = \Delta w$. Then we have for $u \in \mathrm{Ker}(\Delta)$ that

$$(v,u) = (\Delta w, u) = (w, \Delta u) = 0$$

completing the proof. □

Now, consider the following model situation:

Theorem 4. *Let*

$$\begin{tikzcd} U \arrow[r, bend left, "d"] & V \arrow[l, bend left, "d^*"] \arrow[r, bend left, "d"] & W \arrow[l, bend left, "d^*"] \end{tikzcd}$$

be a sequence of finite dimensional inner product spaces such that $d^2 = 0$ and d is formally adjoint to d^* in the sense that

$$(dv, w) = (v, d^* w)$$

for $v \in U, w \in V$ *or* $v \in V, w \in W$*. Let* $\Delta : V \to V$ *equal* $dd^* + d^*d$ *and let* $H = \mathrm{Ker}(\Delta)$*. Then*

$$H \simeq \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*)$$

and V *has a natural orthogonal decomposition as*

$$V \simeq H \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*)$$

with orthogonal decompositions

$$\mathrm{Ker}(d) \simeq H \oplus \mathrm{Im}(d)$$

and

$$\mathrm{Ker}(d^*) \simeq H \oplus \mathrm{Im}(d^*).$$

Thus, there are natural isomorphisms

$$H \simeq \mathrm{Ker}(d)/\,\mathrm{Im}(d) \simeq \mathrm{Ker}(d^*)/\,\mathrm{Im}(d^*).$$

Proof. We first show that $H \simeq \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*)$. Clearly $H \subset \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*)$. For the other inclusion, consider $v \in H$, that is, $\Delta v = 0$. Then we have

$$0 = (\Delta v, v) = (dd^*v, v) + (d^*dv, v) = (d^*v, d^*v) + (dv, dv)$$

which, as desired, implies that $d^*v = dv = 0$ by the non-degeneracy of the inner product.

Now Δ is clearly self-adjoint, so by the Lemma, there is an orthogonal decomposition

$$V \simeq H \oplus \mathrm{Im}(\Delta).$$

So to show that $V \simeq H \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*)$ it suffices to show that

$$\mathrm{Im}(\Delta) \simeq \mathrm{Im}(d) \oplus \mathrm{Im}(d^*).$$

Indeed, it is clear that $\mathrm{Im}(\Delta) \subset \mathrm{Im}(d) + \mathrm{Im}(d^*)$; furthermore,

$$(d\alpha, d^*\beta) = (d^2\alpha, \beta) = 0$$

so $\mathrm{Im}(d) \perp \mathrm{Im}(d^*)$. So we need only show that $\mathrm{Im}(d) \oplus \mathrm{Im}(d^*) \subset \mathrm{Im}(\Delta)$. But $\mathrm{Im}(d) \oplus \mathrm{Im}(d^*) \perp H$ as for $v \in H$ we have

$$(d\alpha + d^*\beta, v) = (\alpha, d^*v) + (\beta, dv) = 0$$

so we must have that $\mathrm{Im}(d) \oplus \mathrm{Im}(d^*) \subset \mathrm{Im}(\Delta)$, as $\mathrm{Im}(\Delta)$ is the orthogonal complement of H by the Lemma.

The rest of the proof follows by noting that $\mathrm{Im}(d^*) \perp \mathrm{Ker}(d)$ as for $v \in \mathrm{Ker}(d)$ we have

$$(v, d^*\alpha) = (dv, \alpha) = 0$$

and that $H \oplus \mathrm{Im}(d) \subset \mathrm{Ker}(d)$ trivially, so we have

$$H \oplus \mathrm{Im}(d) \subset \mathrm{Ker}(d) \subset \mathrm{Im}(d^*)^{\perp} = H \oplus \mathrm{Im}(d)$$

and thus $H \oplus \mathrm{Im}(d) \simeq \mathrm{Ker}(d)$. The statement $\mathrm{Ker}(d^*) \simeq H \oplus \mathrm{Im}(d^*)$ follows identically. □

Now, for a compact Hermitian manifold, $\Gamma(M, \Lambda^q(M))$ and $\Gamma(M, \Omega^{p,q}(M))$ are inner product spaces. To see this, note that if V and W are inner product spaces, $V \otimes W$ naturally has the structure of an inner product space—furthermore, if V is oriented, an inner product on V induces a natural inner product on its wedge powers. So we have Hermitian products $(-,-)_x$ on the fibers $\Lambda^q(M)_x, \Omega^{p,q}(M)_x$; these give an inner product on global sections η, ω via

$$(\eta, \omega)_M = \int_M (\eta, \omega)_x \cdot \mathrm{Vol}_M \, .$$

where Vol_M is the volume form.

Given the analysis above, this inner product suggests that we may be able to answer our question by constructing formal adjoints d^*, (resp. $\overline{\partial}^*$) to the differential d, (resp. $\overline{\partial}$), in the sense that

$$(d\alpha, \beta)_M = (\alpha, d^*\beta)_M \text{ and } (\overline{\partial}\alpha, \beta)_M = (\alpha, \overline{\partial}^*\beta)_M$$

as long as we can deal with the fact that $\Gamma(M, \Lambda^q(M)), \Gamma(M, \Omega^{p,q}(M))$ are infinite-dimensional.

2.2.2 The Hodge Star, d^*, and $\overline{\partial}^*$

Before we define these formal adjoints, we need some preliminaries. Let V be an oriented n-dimensional real vector space with an inner product $(-,-) : V \otimes V \to \mathbb{R}$. Then the inner product induces an isomorphism

$$V \simeq V^*$$

and thus an isomorphism

$$\bigwedge^k(V) \simeq \bigwedge^k(V^*) \simeq \bigwedge^k(V)^*.$$

Furthermore, the orientation on V induces a natural isomorphism

$$\bigwedge^n(V) \simeq \mathbb{R}$$

and thus the wedge product

$$\wedge : \bigwedge^k(V) \otimes \bigwedge^{n-k}(V) \to \bigwedge^n(V) \simeq \mathbb{R}$$

is a perfect pairing, inducing an isomorphism

$$\bigwedge^{n-k}(V) \simeq \bigwedge^k(V)^*.$$

Composing isomorphisms

$$\bigwedge^k(V) \simeq \bigwedge^k(V)^* \simeq \bigwedge^{n-k}(V)$$

gives an isomorphism which we denote $* : \bigwedge^k(V) \simeq \bigwedge^{n-k}(V)$—that is, the Hodge star. Easy computation [18] gives that

$$*^2 = (-1)^{k(n-k)}$$

and that

$$\alpha \wedge *\beta = (\alpha, \beta)\omega$$

where the inner inner product on $\bigwedge^k(V)$ is induced by the inner product on V, and ω is the orientation n-form.

In the cases we are working with, V will have a complex structure $I : V \to V$ with $I^2 = -1$ preserved by the inner product, giving V a natural orientation; $*$ will induce an isomorphism

$$\bigwedge^{k}(V) \underset{\mathbb{R}}{\otimes} \mathbb{C} \simeq \bigwedge^{n-k}(V) \underset{\mathbb{R}}{\otimes} \mathbb{C}$$

which by abuse of notation we will denote by $*$ as well.

If M is a compact Hermitian manifold with complex dimension n, then applying $*$ on the level of fibers gives isomorphisms

$$\Lambda^k(M) \simeq \Lambda^{2n-k}(M)$$

and

$$\Omega^{p,q}(M) \simeq \Omega^{n-p,n-q}(M).$$

Again, we will abuse notation and call these maps $*$. We observe that $\alpha \wedge \overline{*\beta} = (\alpha, \beta)_x \operatorname{Vol}_M$, and thus

$$(\alpha, \beta)_M = \int_M \alpha \wedge \overline{*\beta}.$$

Now, we have by Stokes' Theorem that for $\alpha \in \Lambda^k(M), \beta \in \Lambda^{k+1}(M)$,

$$0 = \int_M d(\alpha \wedge \overline{*\beta}) = \int_M d\alpha \wedge \overline{*\beta} + (-1)^k \int_M \alpha \wedge \overline{d*\beta} = (d\alpha, \beta) + (\alpha, (-1)^k *^{-1} d * \beta).$$

So we may set $d^* : \Lambda^{k+1}(M) \to \Lambda^k(M)$ equal to $(-1)^{k+1} *^{-1} d*$. A similar argument gives $\overline{\partial}^* = - * \partial *$. We let

$$\Delta_d = dd^* + d^*d \text{ and } \Delta_{\overline{\partial}} = \overline{\partial}\, \overline{\partial}^* + \overline{\partial}^* \overline{\partial}.$$

Given a Hermitian vector bundle E with sheaf of holomorphic sections $\mathcal{E}$ we may let $\Delta_E : \Omega^{0,\bullet}(E) \to \Omega^{0,\bullet}(E)$ be given by $\Delta_{\overline{\partial}} \otimes \mathrm{id}$. Then we have the Hodge Theorem:

Theorem 5. *Let $\mathcal{H}^k(M) \subset \Lambda^k(M)$ be the kernel of Δ_d, and let $\mathcal{H}^{0,q}(E) \subset \Omega^{0,q}(M)$ be the kernel of Δ_E; in particular, let $\mathcal{H}^{p,q}(M) = \mathcal{H}^{0,q}(\Omega^p)$. We refer to these spaces as spaces of "harmonic forms." Then the operators Δ_E, Δ_d are elliptic, and there are natural isomorphisms*

$$\mathcal{H}^k(-) \simeq H^k_{DR}(-)$$

and

$$\mathcal{H}^{p,q}(-) \simeq H^{p,q}(-, \mathbb{C}),$$

as well as a canonical isomorphism

$$\mathcal{H}^{0,q}(E) \simeq H^q(M, \mathcal{E}).$$

Composing with the natural inclusions $\mathcal{H}^k(-) \hookrightarrow \Lambda^k(M)$*, etc. give that the sequences (2), (3), as well as the coresponding sequences for* $\mathcal{E}$*, split naturally.*

Proof. (Sketch) The proof is analogous to that of theorem 4, after one shows that the operators Δ_E, Δ_d are elliptic, and applies the following theorem on elliptic operators:

If $P : E \to F$ is an elliptic operator on a compact manifold with E, F of equal rank and equipped with metrics. Then $\ker(P) \subset C^\infty(E)$ is finite dimensional, $P(C^\infty(E)) \subset C^\infty(F)$ is closed of finite codimension, and

$$C^\infty(E) \simeq \ker(P) \oplus P^*(C^\infty(F))$$

orthogonally in the L^2 metric. [18]

Then the isomorphism are given by the inclusion of harmonic forms into the kernel of d. □

As expected, this theorem has important consequences, most notably the following difficult theorem on the sheaf cohomology of holomorphic vector bundles:

Corollary 1. *The de Rham and Dolbeault cohomology groups, and indeed the sheaf cohomology groups of the sheaf of holomorphic forms for any holomorphic vector bundle over a compact Hermitian manifold, are finite dimensional.*

Proof. This follows immediately from the theorem on elliptic operators cited above. □

2.3 Kähler Manifolds

So we have answered our first significant question above—the latter we will answer in the case of Kähler manifolds, an important class of complex manifolds which include complex projective space $\mathbb{P}^n_{\mathbb{C}}$ and smooth complex projective varieties.

Let M be a Hermitian manifold with Hermitian form h. Let

$$\omega = -\operatorname{Im}(h);$$

one may check that this is naturally an element of $\Omega^{1,1}(M) \cap \Lambda^2(M)_{\mathbb{R}}$. We say that M is a Kähler manifold if ω is a closed form. Note that any Kähler form is a symplectic form—that is, all Kähler manifolds are symplectic. Furthermore, note that any complex submanifold of a Kähler manifold is Kähler, as the submanifold inherits by restriction the Hermitian form and associated Kähler form.

Example 1. *The standard Hermitian metric on $\mathbb{C}^n$ induces a Kähler structure.*

Example 2. *Any Hermitian metric on a Riemann surface induces a Kähler structure.*

Proof. The Kähler form is of degree 2—as a Riemann surface is a manifold of real dimension 2, this form is automatically closed. □

Example 3. *Any complex torus is Kähler.*

Proof. Let $T = \mathbb{C}^n/\Gamma$ be a complex torus, where Γ is a discrete, full-rank subgroup of the additive group $\mathbb{C}^n$. Then the standard Hermitian structure (with constant coefficients) on $\mathbb{C}^n$ is clearly Γ-invariant and thus induces a Kähler structure on T. □

Example 4. *Let V be a finite-dimensional complex vector space. Then $\mathbb{P}(V)$, viewed as a complex manifold, is Kähler. Furthermore, any smooth complex submanifold of $\mathbb{P}^n_{\mathbb{C}}$ is Kähler. In particular, smooth projective varieties over $\mathbb{C}$, when viewed as manifolds, are Kähler.*

Proof. See [18, pgs. 76-77]. □

The last point of this example—that smooth projective varieties over $\mathbb{C}$ are Kähler—explains the importance of the Kähler property to mathematicians. Indeed, the property has several remarkable consequences.

2.3.1 Hodge Decomposition and Lefschetz Theorems

The first of these consequences answers our question:

Is there a simple statement of the relationship between $H^{p,q}(M,\mathbb{C})$ and $H^k_{DR}(M)$?

Let $L : \Lambda^k(M) \to \Lambda^{k+2}(M)$ be the operator sending $\alpha \to \omega \wedge \alpha$, where ω is the Kähler form; let K be its formal adjoint with respect to the inner product $(-,-)_M$, that is, $K = (-1)^k * L*$ (by similar arguments to our other adjoint constructions). Let $H : \Lambda^k(M) \to \Lambda^k(M)$ be the operator given by $[L, K]$ where $[-,-]$ is the commutator.

Proposition 1. *The operators L, K, H induce a representation of $\mathfrak{sl}_2$ on $\Lambda^\bullet(M)$, which descends to a representation on $H^*_{DR}(M)$. Furthermore, the action of H on $\Lambda^k(M)$ and thus $H^k_{DR}(M)$ is simply multiplication by $(k-n)$; that is, $H^k_{DR}(M)$ is the weightspace of degree $k-n$.*

Proof. See [18, pg. 145]. The reference does not identify the action as an $\mathfrak{sl}_2$ representation, but it is clear by inspection. $\square$

In particular, if M has complex dimension n, then by the representation theory of $\mathfrak{sl}_2$ we immediately have the following:

Corollary 2 (Hard Lefschetz Theorem)**.** *If $k \leq n$, the map $L^{n-k} : H^k_{DR}(M) \to H^{2n-k}_{DR}(M)$ is an isomorphism.*

Corollary 3 (Lefschetz Decomposition)**.** *Let*

$$H^k_{prim}(M) := \ker(L^{n-k+1} : H^k_{DR}(M) \to H^{2n-k+2}_{DR}(M)).$$

We call this the primitive part of the cohomology of M. Then we have that

$$H^m_{DR}(M) = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} L^k H^{m-2k}_{prim}(M).$$

Note that this decomposition allows one to reconstruct the de Rham cohomology of M from its primitive part.

Proposition 2. *We have*

$$[K, \overline{\partial}] = -i\partial^*, [K, \partial] = i\overline{\partial}^*.$$

Proof. The two propositions are equivalent by taking conjugates; for a proof of the first, see [18, pgs. 139-141]. $\square$

Corollary 4.

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_\partial.$$

Proof. Note first that by proposition 2 above,

$$\partial^*\overline{\partial} = i[K, \overline{\partial}]\overline{\partial} = iK\overline{\partial}^2 - i\overline{\partial}K\overline{\partial} = -i\overline{\partial}K\overline{\partial} = -\overline{\partial}(i[K, \overline{\partial}]) = -\overline{\partial}\partial^*.$$

But then writing $\overline{\partial}^*$ as $-i[K, \partial]$, we have

$$\begin{aligned}
\Delta_d &= (\partial + \overline{\partial})(\partial^* - i[K, \partial]) + (\partial^* - i[K, \partial])(\partial + \overline{\partial}) \\
&= \partial\partial^* + \overline{\partial}\partial^* + i\overline{\partial}\partial K - i\overline{\partial} K \partial + \partial^*\partial + \partial^*\overline{\partial} - iK\partial\overline{\partial} + i\partial K \overline{\partial} \\
&= \partial\partial^* + \partial^*\partial + i\overline{\partial}\partial K - i\overline{\partial} K \partial - iK\partial\overline{\partial} + i\partial K \overline{\partial} \\
&= \partial\partial^* + \partial^*\partial + \partial(i[K, \overline{\partial}]) + i([K, \overline{\partial}])\partial \\
&= 2(\partial\partial^* + \partial^*\partial) \\
&= 2\Delta_\partial
\end{aligned}$$

The other identity follows analogously. ☐

Corollary 5 (Hodge Decomposition)**.** *There is a canonical isomorphism*

$$H^k_{DR}(M) \simeq \bigoplus_{p+q=k} H^{p,q}(M, \mathbb{C}).$$

Furthermore,

$$H^{p,q}(M, \mathbb{C}) \simeq \overline{H^{q,p}(M, \mathbb{C})}.$$

as subspaces of $H^k_{DR}(M)$.

The Hodge decomposition descends to an analogous decomposition on the primitive part of the cohomology.

Proof. This is immediate—we have that a form is d-harmonic if and only if it is $\overline{\partial}$-harmonic. ☐

We may now examine a couple of sample applications

Corollary 6. *If X is Kähler, then* $\dim H^{2k+1}(X, \mathbb{C})$ *is even.*

Proof. We have that

$$\begin{aligned}
\dim H^{2k+1}(X, \mathbb{C}) &= \sum_{p+q=2k+1} \dim H^{p,q}(X) \\
&= \sum_{p+q=2k+1, p<q} \dim H^{p,q}(X) + \dim \overline{H^{p,q}(X)} \\
&= \sum_{p+q=2k+1, p<q} 2 \dim H^{p,q}(X)
\end{aligned}$$

which is clearly even. ☐

Example 5 (Not all compact complex manifolds are Kähler)**.** *Let $\mathbb{Z}$ act on $\mathbb{C}^2 - \{0\}$ via multiplication by 2^n. Then the quotient space H of $\mathbb{C}^2 - \{0\}$ by this action, is compact and is not Kähler.*

Proof. To see compactness, consider a cover $\{U_i\}$ of H, which we may lift to a cover of $\mathbb{C}^2 - \{0\}$ by taking preimages through the quotient map. Then select a finite subset covering $\{(z_1, z_2) \mid |1 \le |z_1^2 + |z_2|^2 \le 2\}$ which exists by compactness of this region—as this region's orbit is all of $\mathbb{C}^2 - \{0\}$, this finite subset gives a finite cover of H.

But now note that $\mathbb{C}^2 - \{0\}$ is simply connected and locally homeomorphic to H, and thus is the universal covering space of H; as H is the quotient by $\mathbb{Z}$ acting properly discontinuously, the fundamental group of $\pi_1(H)$ is $\mathbb{Z}$. But then by the Hurewicz theorem, $H^1(H, \mathbb{Z}) = \mathbb{Z}$, and thus $H^1(H, \mathbb{C}) = H^1(H, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is one-dimensional. But then H cannot be Kähler by Corollary 6 for $X = H, k = 0$. $\square$

Example 6 (Computing the Dolbeault Cohomology of projective space)**.** *Let V be a complex vector space of complex dimension n. Then we claim that*

$$H^{p,q}(\mathbb{P}(V), \mathbb{C}) \simeq 0$$

for $p \neq q$, or $p + q > 2n$ and

$$H^{p,p}(\mathbb{P}(V), \mathbb{C}) \simeq \mathbb{C}$$

for $p < n$.

Proof. It is well known that the cohomology ring $H^*(\mathbb{P}(V), \mathbb{C}) = \mathbb{C}[t]/t^{n+1}$ where t is of degree 2. In particular, $\dim H^{2k}(\mathbb{P}(V), \mathbb{C}) = 1$ for $k < n$ and $\dim H^{2k+1}(\mathbb{P}(V), \mathbb{C}) = 0$ for all k. As $\mathbb{P}(V)$ is Kähler, we have the Hodge decomposition

$$H^m(\mathbb{P}(V), \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(\mathbb{P}(V), \mathbb{C}).$$

In particular, we have for $p \neq q$, with $p + q$ odd, $H^{p,q}(\mathbb{P}(V), \mathbb{C})$ must be zero immediately. For $p + q$ even, we have $\dim H^{p,q}(\mathbb{P}(V), \mathbb{C}) = \dim H^{q,p}(\mathbb{P}(V), \mathbb{C})$ and thus

$$2 \dim H^{p,q}(\mathbb{P}(V), \mathbb{C}) \le \dim H^{p+q}(\mathbb{P}(V), \mathbb{C}) = 1$$

so $H^{p,q}(\mathbb{P}(V),\mathbb{C}) = 0$. Combining this information with

$$1 = \dim H^{2k}(\mathbb{P}(V),\mathbb{C}) = \sum_{p+q=2k} \dim H^{p,q}(\mathbb{P}(V),\mathbb{C})$$

gives $\dim H^{k,k}(\mathbb{P}(V),\mathbb{C}) = 1$ and completes the proof. $\square$

Remark 1. *The Hodge decomposition is independent of the Kähler structure. Indeed it is dependent only on $\Delta_d, \Delta_{\overline{\partial}}$, both of which depend only on the complex structure.*

3 Mixed Hodge Structure and Variations of Structure

Let us abstract the situation of the Hodge decomposition somewhat, and then extend it to a more general case. Our motivation here and in the next chapter is to define a structure on cohomology that exists for a family $f : E \to B$ of complex varieties over a (smooth) base; that is, f is a proper holomorphic submersion. In particular, while the generic fiber should be smooth, the locus of points of B whose fibers are singular may form a non-empty proper subvariety of B.

3.1 Hodge Structure

Consider first the case where B is a point and E is Kähler. Then we have the integral cohomology of E, given by $H^k(E,\mathbb{Z})$, and the Hodge decomposition above gives structure to $H^k(E,\mathbb{C}) \simeq H^k(E,\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{C}$.

In general, we may let A be a finitely generated free Abelian group embedded in a real vector space $V_{\mathbb{R}}$, and let $V_{\mathbb{C}} = V_{\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}$. The data of a *Hodge structure of degree* n on $(A, V_{\mathbb{R}}, V_{\mathbb{C}})$ is a collection $\{V^{p,q}\}$ and a decomposition

$$V_{\mathbb{C}} \simeq \bigoplus_{p+q=n} V^{p,q}$$

with

$$V^{p,q} \simeq \overline{V^{q,p}}.$$

Another point of view will become important in the sequel. Let

$$F^p V = \bigoplus_{s \geq p} V^{s,n-s}$$

be a decreasing filtration of $V_{\mathbb{C}}$.

Then we have that

$$V^{p,q} = F^p V \cap \overline{F^q V},$$

so the filtration determines the Hodge decomposition. Furthermore, for all p we have

$$V_{\mathbb{C}} = F^p V \oplus \overline{F^{n-p+1} V}.$$

Indeed, it is easy to see that any filtration satisfying this last property induces a Hodge structure on A.

Let V' be a Hodge structure on A', of degree m. Note that $\mathrm{Hom}(V, V')$ inherits a natural Hodge structure of degree $m-n$; $s \in \mathrm{Hom}(V, V')$ is of type p, q if $s(V^{s,r}) \subset {V'}^{s+p,r+q}$ for all s, r. As a special case V^* has a natural Hodge structure of degree $-n$, where we let $V' = \mathbb{C}$ with the trivial Hodge structure of degree 0. Through the isomorphism $V \otimes V' \simeq \mathrm{Hom}(V^*, V')$, tensor products obtain a natural Hodge structure of degree $n+m$.

Furthermore, Hodge structures naturally form a category; a morphism $V \to V'$ of type (r,r) is a linear map $V \to V'$ defined over $\mathbb{Q}$ (relative to the natural inclusions $A \to V, A' \to V'$) which is of type (r,r) relative to the Hodge structure on $\mathrm{Hom}(V, V')$; a morphism of Hodge structures is a morphism of type $(0,0)$.

3.1.1 Polarized Hodge Structures

There is some additional structure on the cohomology of a Kähler manifold which is not contained in its Hodge decomposition—as we noted above, the Hodge decomposition is independent of the Kähler structure. However, the Kähler structure does induce further structure on the cohomology. Indeed, there is a *polarization* on the Hodge structure of the cohomology of a Kähler manifold.

We say a Hodge structure $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ is polarized by Q if Q is a bilinear form on V defined over $\mathbb{Q}$ (relative to A). We require that Q be symmetric if n is even, skew if n is odd, that $V^{p,q}$ be perpendicular to $V^{p',q'}$ unless $p = p', q = q'$, and we require that $i^{p-q} Q(v, \overline{v}) > 0$ if $v \in V^{p,q}$ is nonzero.

To see this structure on a Kähler manifold, we note that the Hodge structure on $H^*_{DR}(M)$ induces a Hodge structure on the primitive part of the cohomology—for $\alpha, \beta \in H^k_{prim}(M)$ with $k \leq \dim_{\mathbb{C}}(M)$ we have that

$$Q(\alpha, \beta) = \int_M \omega^{n-k} \wedge \alpha \wedge \beta$$

satisfies the desired properties.

Remark 2. *We have here defined* rational *morphisms and polarizations; similarly, we may consider integral* *morphisms and polarizations, which are defined over $\mathbb{Z}$ relative to the lattice $A \subset V$, rather than over $\mathbb{Q}$.*

3.2 Mixed Hodge Structure

Unfortunately, Hodge theory does not suffice for algebra-geometric applications, as a Hodge structure does not exist for the cohomology of non-compact (quasi-projective), possibly singular varieties over $\mathbb{C}$. A theorem of Deligne gives an extension of the Hodge decomposition for this situation.

A *mixed Hodge structure* on a finitely generated free Abelian group $A \subset V_{\mathbb{R}}$ consists of an increasing filtration

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{m-1} \subset W_m \subset W_{m+1} \subset \cdots \subset V$$

on $V_{\mathbb{C}} = V_{\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}$ and a decreasing filtration

$$V \supset \cdots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \cdots \supset F_0 = 0$$

such that the filtration F^* induces a Hodge structure on $Gr_m(W_*) := W_m/W_{m_1}$ of weight m, via

$$F^p(Gr_m(W_*)) := (W_m \cap F^p)/(W_{m-1} \cap F^p).$$

The filtration W_* is referred to as the weight filtration, and the filtration F^* is the Hodge filtration.

Note that the Hodge structure is a special case of mixed Hodge structure, where $W_m = V$ and $W_i = 0$ for $i < m$. Furthermore, if $H^*_{DR}(M)$ is the de Rham cohomology of a Kähler manifold, then setting

$$W_i = \bigoplus_{j \leq i} H^j_{DR}(M)$$

and

$$F^p = \bigoplus_{p' \geq p} H^{p,q}(M)$$

gives a mixed Hodge structure on $H^*_{DR}(M)$.

A morphism of mixed Hodge structures is, as before, a linear map $f : V \to V'$ defined over $\mathbb{Q}$ relative to the lattices within V, V' such that the weight and Hodge filtrations are preserved, e.g. $f(W_m) \subset W'_m$ and $f(F^p) \subset F'^p$.

Remark 3. *One may check that the category of mixed Hodge structures with morphisms given by morphisms* *of mixed Hodge structures is an Abelian category. [7]*

The importance of mixed Hodge structures stems initially from the following theorem of Deligne:

Theorem 6. *Let X be a quasiprojective variety over $\mathbb{C}$. Then $H^*(X)$ has a functorial mixed Hodge structure. [2-7]*

3.3 Variations of Structure

We now consider the more general case of a family of varieties. Consider a proper holomorphic submersion $f : E \to B$ with smooth connected fibers, where E, B are complex manifolds. Assume E is embedded in $\mathbb{P}^n_{\mathbb{C}}$ for some n (fix such an embedding), but is not necessarily closed. Then for each $b \in B$, $X_b := f^{-1}(b)$ is a complex, connected projective manifold. We wish to consider the sheaves $R^m_{f_*}(\mathbb{C})$, where here $\mathbb{C}$ is the constant $\mathbb{C}$-valued sheaf on E. In the case that B is just a point, this is the Abelian group $H^k(E, \mathbb{C}) \simeq H^k_{DR}(E)$, in which we have a (mixed) Hodge decomposition. We claim that we find a similar decomposition in the general case.

One may check that this sheaf is the sheaf of flat sections of a flat complex vector bundle H^m over B (this follows by considering the map f as giving a C^∞ vector bundle over S, and considering the cohomology on local trivializations).

Again considering local trivializations, one finds that the map L defined for Kähler manifolds extends to a map of vector bundles $H^m \to H^{m+2}$; as before, we may set P^k to be the kernel of $L^{n-k+1} \subset H^k$ (where $n = \dim_{\mathbb{C}}(M)$), the Hodge theorem applied to local trivializations gives a polarized Hodge structure F^p on P^k (by which we mean a bilinear form inducing a polarized Hodge structure on each fiber).

A technical condition, important for arguments about this sort of situation, is that

$$\nabla \mathcal{O}(F^p) \subset \mathcal{O}(F^{p-1} \otimes T^*)$$

where ∇ is the flat connection on P^k. This is referred to as Griffiths transversality.

We may abstract this case to a general definition. A *variation of Hodge structure* on the base B, of weight m, consists of a flat complex vector bundle H over B which is the complexification of a real bundle with an embedded flat lattice bundle A; a flat bilinear form defined over $\mathbb{Q}$ skew if m is odd and symmetric otherwise; and a descending filtration of H given by (F^p) satisfying

$$\nabla \mathcal{O}(F^p) \subset \mathcal{O}(F^{p-1} \otimes T^*).$$

Furthermore, the (F^p) must define a Hodge structure of weight m on each fiber of H, which is polarized by Q.

Of course, one may define an analogous variation of mixed Hodge structure, as well as morphisms of such variations; these definitions are entirely analogous to those in the previous two sections, so we omit them.

3.4 The Period Mapping

There is another important viewpoint on variations of Hodge structure [1], which we will examine in more detail—in particular, we may consider variations of Hodge structure to be pullbacks of a universal bundle over a classifying space for Hodge structures. In this section we elucidate the construction of this classifying space, and the correspondence between certain maps into the classifying space and variations of Hodge structure.

We consider Hodge structures $(V, A, \{V^{p,q}\})$ of degree n, satisfying $\dim(V^{p,q}) = h^{p,q}$ for fixed integers $h^{p,q}$ (with $\sum h^{p,q} = \dim_{\mathbb{C}}(V)$, $h^{p,q} = h^{q,p}$, and $h^{p,q} = 0$ if $p + q \neq n$). The set of all decreasing filtrations F^p of V with $\dim F^p = \sum_{j \geq p} h^{j,n-j}$ is a complex projective variety.

To see this, consider a product of Grassmannians

$$\prod_p \mathbb{G}(\sum_{j \geq p} h^{j,n-j}, V).$$

Then a tuple of points (F^i) is a filtration (e.g. $\cdots \supset F^i \supset F^{i+1} \supset \cdots$) etc. if and only if it is in the vanishing locus of each of the map $\mathcal{M}_{i+1} \to V \otimes \mathcal{O}/\mathcal{M}_i$ where $\mathcal{M}_i$ is the canonical vector bundle over $\mathbb{G}(\sum_{j \geq i} h^{j,n-j}, V)$ and the map is induced by the natural inclusion $\mathcal{M}_{i+1} \to V \otimes \mathcal{O}$. It is easy to see that $\mathrm{GL}_{\mathbb{C}}(V)$ acts transitively and holomorphically on the variety of filtrations, so it is a nonsingular complex projective variety, denoted $\hat{\mathcal{F}}$. Now those filtrations satisfying $V = F^p V \oplus \overline{F^{n-p+1}} V$ for all p form an open subset of this variety (in the Hausdorff topology), as two subspace intersecting non-trivially is a closed condition; denote this complex manifold by $\mathcal{F}$. This space classifies Hodge structures with $\dim V^{p,q} = h^{p,q}$.

One may fix a non-degenerate bilinear form S, skew if n is odd and symmetric of n is even, and construct a similar classifying space for Hodge structures polarized by this form. These spaces have the advantage that they are homogeneous spaces for certain Lie groups. In particular, let $\hat{D} \subset \hat{\mathcal{F}}$ be the set of filtrations such that $S(F^p, \overline{F^{n-p+1}}) = 0$; this is a subvariety. Let $D \subset \hat{D}$ be the subset of filtrations satisfying the other condition of polarization, that is, $i^{p-q} S(v, \overline{v}) > 0$ for $v \in H^{p,q}, v \neq 0$. It is easy to see that $D \subset \mathcal{F}$.

Fix A a lattice in a real vector space $V_{\mathbb{R}}$ with complexificaiton $V_{\mathbb{C}}$; we wish to classify Hodge structures on $(A, V_{\mathbb{R}}, V_{\mathbb{C}})$. Now let $G_{\mathbb{C}}$ be the subgroup of $GL_{\mathbb{C}}(V_{\mathbb{C}})$ preserving S, and $G_{\mathbb{R}}$ the subgroup of $GL_{\mathbb{R}}(V_{\mathbb{R}})$ preserving S. We have that $G_{\mathbb{C}}$ acts transitively on $\hat{D}$ and $G_{\mathbb{R}}$ acts transitively on D, so we have that $\hat{D}, D$ are homogeneous spaces for $G_{\mathbb{C}}, G_{\mathbb{R}}$.

Let $G_{\mathbb{Z}}$ be the subset of $G_{\mathbb{R}}$ consisting of elements g with $gA = A$. It is not hard to see that the action of $G_{\mathbb{Z}}$ on D is properly discontinuous, so for any discrete subgroup of $G_{\mathbb{Z}}$, the quotient of D by Γ, denoted Γ/D, gives a complex analytic variety.

Now we define a subbundle of the tangent bundle to $\hat{D}$, $T_h(\hat{D})$, by saying that a vector v in the tangent bundle at x is in $T_h(\hat{D})$ if, viewed as an endomorphism of $V_{\mathbb{C}}$, it sends $F^p(x)$ into $F^{p-1}(x)$, where these are the members of the filtrations at the point x, for all p. By restricting to D, this defines an analogous subbundle for D. We say a map into $D, \hat{D}$ is horizontal if its differential takes values in $T_h(\hat{D})$, $T_h(D)$.

Now consider a variation of Hodge structures $\mathbb{H}$ over M. Pulling back to a bundle over the universal cover $\tilde{M}$ of M, we have that as $\tilde{M}$ is simply connected, we get a trivial bundle $H' \times \tilde{M}$ where H' is a complex vector space. Furthermore, the filtration of $\mathbb{H}$ pulls back to a filtration of $H' \times \tilde{M}$, such that the filtration induces a Hodge structure on each fiber. If $\mathbb{H}$ was polarized, we may pull back the polarization form to a form on H', polarizing each fiber, so we have a horizontal map $\tilde{M} \to D$ sending a point to the Hodge structure on H' over that point.

Realizing $\mathbb{H}$ as a quotient of $H' \times \tilde{M}$ by an action of $\pi_1(M)$, we have that π_1 acts on H' preserving the lattice, and thus maps into $G_{\mathbb{Z}}$. Let its image by Γ. Then we have a map $M \to \Gamma/D$, which locally lifts to the map $\tilde{M} \to D$, and is holomorphic. This is the “period mapping” of Griffiths.

Note that D has a natural filtered trivial bundle H with the fiber over a point $p \in D$ given by the Hodge stucture at that point. Pulling back this bundle through the map $\tilde{M} \to D$ gives exactly the trivial bundle $H' \times \tilde{M}$ with the filtration above—this is the sense in which this natural bundle H is universal.

4 Mixed Hodge Modules

We now consider a further generalization of this situation, due to Morihiko Saito [12-15, 17]. This chapter assumes familiarity with the definitions of $\mathcal{D}_X$-modules and perverse sheaves [17].

4.1 Hodge Modules

Let X be a smooth complex projective variety. We say that an increasing filtration F^p of a $\mathcal{D}_X$-module is good if it respects the action of $\mathcal{D}_X$ and if $\mathrm{Gr}^F(M)$ is coherent as a $\mathrm{Gr}(\mathcal{D}_X)$ module (relative to the standard filtration on $\mathcal{D}_X$). We say that a filtered $\mathcal{D}_X$-module is holonomic if the degree of the Hilbert polynomial of M (its dimension) is equal to the dimension of X as a complex variety. Equivalently, we say that it is holonomic if, viewed as an object of the derived category $\mathbb{D}(M)$ has cohomology concentrated in dimension 0 (where $\mathbb{D}$ is the Verdier dual functor).

Let $MF_h(\mathcal{D}_X)$ be the category of holonomic $\mathcal{D}_X$-modules with a good filtration, with morphisms given by morphisms of filtered $\mathcal{D}_X$-modules. If k is a field, we may let k_X be the constant k-valued sheaf over X, and $\mathrm{Perv}(k_X)$ the category of perverse sheaves over k_X. There is a functor

$$DR : \mathcal{D}_X - \mathrm{mod} \to \mathbb{C}_X - \mathrm{mod}$$

given by tensoring with $\mathcal{O}_X$ over $\mathcal{D}_X$; this induces a functor

$$DR : MF_h(\mathcal{D}_X) \to \mathrm{Perv}(\mathbb{C}_X).$$

Furthermore, there is a functor

$$- \underset{\mathbb{Q}}{\otimes} \mathbb{C} : \mathrm{Perv}(\mathbb{Q}_X) \to \mathrm{Perv}(\mathbb{C}_X).$$

One may check that both of these functors are fully faithful. Let $MF_h(\mathcal{D}_X, \mathbb{Q})$ be the pushout category

$$MF_h(\mathcal{D}_X) \underset{\mathrm{Perv}(\mathbb{C}_X)}{\times} \mathrm{Perv}(\mathbb{Q}_X).$$

That is, the objects are triples $((M, F^p), L, \alpha)$, where $(M, F^p) \in MF_h(\mathcal{D}_X), L \in \mathrm{Perv}(\mathbb{Q}_X)$, and α is an isomorphism $DR(M) \simeq \mathbb{C} \underset{\mathbb{Q}}{\otimes} L$, and morphisms are pairs of morphisms compatible with α—that is, (f, f') such that the diagram

$$\begin{tikzcd} DR(M) \arrow[r, "\alpha"] \arrow[d, "DR(f)"] & \mathbb{C} \underset{\mathbb{Q}}{\otimes} L \arrow[d, "\mathbb{C} \underset{\mathbb{Q}}{\otimes} f'"] \\ DR(M') \arrow[r, "\alpha'"] & \mathbb{C} \underset{\mathbb{Q}}{\otimes} L' \end{tikzcd}$$

commutes.

Let $MF_h(\mathcal{D}_X, \mathbb{Q})_{str}$ be the full subcategory of $MF_h(\mathcal{D}_X, \mathbb{Q})$ consisting of direct sums of objects $((M, F), L, \alpha)$ of $MF_h(\mathcal{D}_X, \mathbb{Q})$ such that no sub-object or quotient object has support strictly smaller than that of $((M, F), L, \alpha)$ (which we identify with the support of M or L).

Now we inductively define the category $MH(X, n)$ of Hodge modules over X of weight n, where X is a smooth complex algebraic variety. It is the full subcategory of $MF_h(\mathcal{D}_X, \mathbb{Q})_{str}$ satisfying the following conditions.

- If $((M,F),K,\alpha)$ is supported on a point $\{x\}$, then it is the direct image of a rational Hodge structure of weight n $(A, V, (V^{p,q}))$ through the inclusion $i_x : x \to X$, where the direct image is viewed as acting on $\mathcal{D}_X$-modules and perverse sheaves, and the Hodge structure is thought of as a member of $MF_h(\mathcal{D}_x, \mathbb{Q})_{str}$. In particular, the category of Hodge modules of weight n over a point is exactly the category of rational Hodge structures of weight n.

- For all Zariski open $U \subset X$ and t a non-constant holomorphic function on U, an object $\mathcal{M}$ of $MH(X,n)$, if, not supported on $t^{-1}(0)$, must satisfy that the k-th graded component of $\psi_t((M,F),K), \phi_{t,1}((M,F),K)$ with respect to the weight filtration are Hodge modules of weight k with support on $t^{-1}(0)$, where $\psi_t, \phi_{t,1}$ are defined in [12-15] and the weight filtration is defined in [17]. (The definitions are extremely technical, so we omit them.)

As before, we may require that these objects be polarizable, in which case we denote the category by $MH(X,n)^{(p)}$.

The important thing to note is that polarizable variations of Hodge structure are naturally Hodge modules; indeed, we have that

Theorem 7. *Let X be a smooth complex variety of dimension d_X and $H, (F^p)$ a variation of Hodge structure of weight n on X. Then there is a connection ∇ on $H \underset{\mathbb{Q}}{\otimes} \mathcal{O}_X$ such that*

$$(((H \underset{\mathbb{Q}}{\otimes} \mathcal{O}_X, \nabla), (F^{-p})), H[d_X])$$

is a Hodge module of weight $n + d_X$.

4.2 Mixed Hodge Modules

We now continue the road to generalization. Following Peters and Steenbrink [17], we give an axiomatic definition of a Mixed Hodge module, which generalizes the concept of a variation of mixed Hodge structure.

Let $MHW(X)^{(p)}$ be the category of polarizable W-filtered Hodge modules—that is, objects are W-filtered elements $((M,F),L,W)$ of $MF_h(X,\mathbb{Q}_X)$ such that $\mathrm{Gr}_i^W((M,F),L)$ is an element of $MH(X,i)^{(p)}$ for all i. Then there is a full Abelian subcategory of $MHW(X)^{(p)}$ satisfying the following conditions.

- Let $D^b(MHM(X))$ be the derived category of bounded complexes in $MHM(X)$, and let $D^b_{cs}(X;\mathbb{Q})$ be the derived category of sheaves of vector spaces over $\mathbb{Q}$ whose cohomology sheaves are constructible. Then there is a faithful functor

$$\mathrm{rat}_X : D^bMHM(X) \to D^b_{cs}(X;\mathbb{Q})$$

sending $MHM(X)$ to $\mathrm{Perv}(X;\mathbb{Q})$, the category of perverse sheaves of $\mathbb{Q}$-vector spaces.

- There is a faithful functor

$$\mathrm{Dmod}_X : D^bMHM(X) \to D^b_{coh}(\mathcal{D}_X),$$

where $D^b_{coh}(\mathcal{D}_X)$ is the derived category of $\mathcal{D}_X$-complexes with coherent cohomology.

- Let $DR_X : D^b_{coh}(\mathcal{D}_X) \to D^b_{cs}(X;\mathbb{C})$ be the functor $- \otimes_{\mathcal{D}_X} \mathcal{O}_X$; then we ask that $\mathrm{rat}_X(-) \otimes \mathbb{C}$ is naturally isomoprhic to $DR_X \circ \mathrm{Dmod}_X$.

- The category of mixed Hodge modules on a point is naturally isomorphic to the category of graded polarizable rational mixed Hodge structures, as defined above; rat_X in this case sends a mixed Hodge structure to its rational vector space.

- Applying the functor Gr_i^W commutes with the cohomology functor H^j.

- The Verdier dualtiy functor $\mathbb{D}_X$ on $D^b_{cs}(X;\mathbb{Q})$ lifts to a functor $\mathbb{D}_X : MHM(X) \to MHM(X)$ commuting with rat_X.

- If $f : X \to Y$ is a morphism of varieties, there are functors $f_*, f_! : D^bMHM(X) \to D^bMHM(Y), f^*, f^! : D^bMHM(Y) \to D^bMHM(X)$ satisfying the usual properties with regard to adjointness and $\mathbb{D}_X$, and commuting with rat_X in the sense that they lift the corresponding functors in $D^b_{cs}(X;\mathbb{Q})$.

- If $\mathrm{Gr}_i^W H^j(M^\bullet)$ is zero for all $i - j > n$, then the same is true for $f_!M^\bullet, f^*M^\bullet$.

- If $\mathrm{Gr}_i^W H^j(M^\bullet)$ is zero for all $i - j < n$, then the same is true for $f^!M^\bullet, f_*M^\bullet$.

Again, variations of mixed Hodge structure are naturally mixed Hodge modules.

These axioms reveal the importance of this generalization—we may now use the four operations $f_*, f^*, f_!, f^!$ of Grothendieck to analyze Hodge structures. That is, we have the full functorial power analogous to sheaf theory at our disposal.

However, we still lack the tools to analyze variations of structure.

5 Schmid's Nilpotent Orbit Theorem and SL_2-Orbit Theorems

5.1 Motivation and Setup

We take the point of view of section 3.4. Fix a real vector space $V_{\mathbb{R}}$ containing a lattice A, and let $V_{\mathbb{C}}$ be its complexification. Let n be an integer, S be a non-degenerate skew (resp. symmetric) bilinear form if n is odd (resp. even), and let D be the classifying space of Hodge structures of degree n on $(A, V_{\mathbb{R}}, V_{\mathbb{C}})$ polarized by S. Let $\Gamma \subset G_{\mathbb{Z}}$ be a subgroup and $\Phi : M \to \Gamma/D$ a map from a complex manifold M which is holomorphic and lifts locally to horizontal maps $\tilde{M} \to D$, where $\tilde{M}$ is the universal cover of M. Furthermore, we assume M is Zariski open in a reduced analytic space $\bar{M}$.

We wish to analyze the possibility of extending Φ to a (possibly singular map) on the ambient space in which M resides. In particular, we consider the case where $\bar{M} - M$ is codimension 1. By work of Hironaka, we may assume $\bar{M} - M$ has only normal crossings. As everything is local, we may thus assume that $M \simeq (B - 0)^l \times B^{k-l}$ and $\bar{M} \simeq B^k$, where B is the unit disk in $\mathbb{C}$.

We only consider the case where $M, \bar{M}$ are one-dimensional, e.g. by passing to the local situation $M \simeq B - 0$ and $\bar{M} \simeq B$. For the generalized situation, see [1]. Then the universal covering space of M is the upper half plane $U := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ via the map $z \mapsto e^{2\pi i z}$. The fundamental group $\pi_1(M) = \mathbb{Z}$ acts on U via transation, e.g. the generator acts via the deck transformation $z \mapsto z + 1$. By the properties of the universal covering space, a locally liftable map $\Phi : M \to \Gamma/D$ lifts to a map $\tilde{\Phi} : U \to D$ such that the diagram

$$\begin{tikzcd} U \arrow[r, "\tilde{\Phi}"] \arrow[d] & D \arrow[d] \\ M \arrow[r, "\Phi"] & \Gamma/D \end{tikzcd}$$

commutes.

5.2 Nilpotent Orbit Theorem

Now again by the properties of covering spaces, there is an element γ of $\Gamma \subset G_{\mathbb{Z}}$, not necessarily unique, such that $\tilde{\Phi}(z + 1) = \gamma \circ \tilde{\Phi}(z)$. Fix such a γ; we will refer to it as the monodromy transformation of Φ; to compute such an example of such a γ in this case, one may simply consider the image of a generator of $\pi_1(M)$ in $G_{\mathbb{Z}}$. A lemma of Borel gives that all the eigenvalues of γ are roots of unity.

Now we use the Jordan decomposition theorem to write $\gamma = \gamma_s \gamma_u$, where γ_s is semisimple and γ_u is unipotent. As all the eigenvalues of γ are roots of unity, γ_s has finite order m. Let

$$N = \log(\gamma_u) = \frac{1}{m}\log(\gamma^m).$$

as an element of the Lie algebra of $G_{\mathbb{R}}$ (whose complexification is the Lie algebra of $G_{\mathbb{C}}$). To see that N exists, we may write

$$N = \sum_{k\geq 1}(-1)^{k+1}\frac{1}{k}(\gamma_u - 1)^k$$

which is a finite sum as γ_u is unipotent. Now remembering that $G_{\mathbb{C}}$ acts on $\hat{D}$ and that $D \subset \hat{D}$, we may define a map $\tilde{\Psi} : U \to \hat{D}$, by

$$\tilde{\Psi}(z) = \exp(-mzN)\cdot\tilde{\Phi}(mz)$$

which satisfies

$$\tilde{\Psi}(z + 1/m) = \exp(-mzN)\gamma_u^{-1}\gamma\cdot\tilde{\Phi}(mz) = \gamma_s\cdot\tilde{\Psi}(z)$$

and thus

$$\tilde{\Psi}(z+1) = \tilde{\Psi}(z).$$

Thus $\tilde{\Psi}$ induces a map $\Psi : M \to \hat{D}$.

The nilpotent orbit theorem gives data on this new map:

Theorem 8 (Nilpotent Orbit Theorem, One-Variable)**.** *The mapping Ψ has a holomorphic continuation to B. Let $a = \Psi(0) \in \hat{D}$. Then a is a fixed point of γ_s, and there exists $\alpha, \beta \geq 0$ such that $\operatorname{Im}(z) > \alpha$ implies $\exp(zN)\cdot a \in D$, and*

$$d(\exp(zN)\cdot a, \tilde{\Phi}(z)) \leq (\operatorname{Im} z)^{\beta} e^{-2\pi m^{-1}\operatorname{Im}(z)}$$

where d is a $G_{\mathbb{R}}$-invariant Riemannian metric on D. Furthermore, $z \mapsto \exp(zN)\cdot a$ is a horizontal map into $\hat{D}$. [1]

In particular, this shows that for $\operatorname{Im}(z)$ large (that is, for points in U which are map to points close to the puncture of M), the map $\tilde{\Phi}(z)$ is close to a special kind of map, that is, $\exp(zN)\cdot a$—which also maps horizontally into D in some neighborhood of the puncture.

5.3 SL_2-Orbit Theorem

We wish to analyze these special maps. First, note that the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$ has a Hodge structure. Indeed, picking a point of D, the Hodge structure on this point induces a Hodge structure on $\mathrm{Hom}(V_{\mathbb{C}}, V_{\mathbb{C}})$; by restriction, this induces a Hodge structure of weight 0 on $\mathfrak{g}$, which we will denote $\{\mathfrak{g}^{p,-p}\}$.

Now assume we are in the situation of the Nilpotent Orbit Theorem—that is, we have a point $a \in \hat{D}$ $\alpha \geq 0$, and N nilpotent in the Lie algebra of $G_{\mathbb{R}}$ such that $z \mapsto \exp(zN) \cdot a$ is horizontal as a map into $\hat{D}$ and, for $\mathrm{Im}(z) > \alpha$, has image in D.

Essentially, the idea here is to analyze the case where D is the upper half plane, in which case $G_{\mathbb{R}} = SL(2, \mathbb{R})$ and $\hat{D}$ is the Riemann sphere $\mathbb{P}^1$.

Indeed, let L be the stabilizer in $SL(2, \mathbb{C})$ of the point $i \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$; then viewing $SL(2, \mathbb{R}) \subset SL(2, \mathbb{C})$, we have that the orbit of i is the upper half plane U. Let D be a classifying space for Hodge structures with base point x; then a homomorphism

$$\psi : SL(2, \mathbb{C}) \to G_{\mathbb{C}}$$

sending L into the stabilizer of x in $G_{\mathbb{C}}$, gives a holomorphic, equivariant map

$$\tilde{\psi} : \mathbb{P}^1 \to \hat{D}$$

via $\tilde{\psi}(g \cdot i) = \psi(g) \cdot x$.

Now let

$$Z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, X_+ = \frac{1}{2} \begin{pmatrix} = i & 1 \\ 1 & i \end{pmatrix}, X_- = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$$

span $\mathfrak{sl}(2, \mathbb{C})$; then letting the span of X_+, Z, X_- have degrees $(-1, 1), (0, 0)$, and $(1, -1)$ respectively, we give $\mathfrak{sl}(2, \mathbb{C})$ a Hodge structure of degree 0 relative to the real vector space $\mathfrak{sl}(2, \mathbb{R})$. The SL_2-orbit theorem states the following:

Theorem 9 (SL_2-orbit theorem)**.** *Let $D, \hat{D}$ be a classifying spaces for Hodge structures on $(A, V_{\mathbb{R}}, V_{\mathbb{C}})$ polarized by S as in section 3.4. Consider $a \in \hat{D}$ $\alpha \geq 0$, and N nilpotent in the Lie algebra of $G_{\mathbb{R}}$ such that $z \mapsto \exp(zN) \cdot a$ is horizontal as a map into $\hat{D}$ and, for $\mathrm{Im}(z) > \alpha$, has image in D. Then there exist*

1. *A homomorphism $\psi : SL(2, \mathbb{C}) \to G_{\mathbb{C}}$.*

2. *A neighborhood W of $\infty \in \mathbb{P}^1$ and a holomorphic map $g : W \to G_{\mathbb{C}}$*

such that

1. $\exp(zN)\cdot a = g(-iz)\tilde{\psi}(z)$ *for* $z \in W - \{\infty\}$*;*

2. $\psi(SL(2,\mathbb{R})) \subset G_{\mathbb{R}}$*, and* $\tilde{\psi}(U) \subset D$*, where* U *is the upper half plane in* $\mathbb{P}^1$*;*

3. $d\psi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$ *is a map of Hodge stuctures of degree* $(0,0)$*;*

4. $g(y) \in G_{\mathbb{R}}$ *for* $iy \in W \cap i\mathbb{R}$*;*

5. $\operatorname{Ad} g(\infty)^{-1}(N)$ *is equal to*
$$d\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

6. *For* $iy \in W \cap i\mathbb{R}$*, let*
$$h(y) = g(y)\exp\left(-\frac{1}{2}\log y \cdot d\psi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$
Then
$$h(y)^{-1}\frac{d}{dy}h(y) \in (\mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}) \cap \mathfrak{g}_{\mathbb{R}}.$$

7. *The endomorphism of* $V_{\mathbb{C}}$ *given by*
$$T = d\psi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
operates semi-simply and has integral eigenvalues. Furthermore, writing the power series expansions of $g(z), g(z)^{-1}$ *about* $z = \infty$ *as*
$$g(z) = g(\infty)(1 + \sum_{i\geq 1} g_i z^{-i}), g(z)^{-1} = g(\infty)^{-1}(1 + \sum_{i\geq 1} f_i z^{-i})$$
we have that f_n, g_n *map the* l*-eigenspace of* T *into the linear span of the* k*-eigenspaces for* $k \leq l+n-1$*.*

Furthermore, by altering the base point of D *if necessary, one may choose* g, ψ *such that* $g(\infty) = 1$ *and*
$$N = d\psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$
then we have
$$(\operatorname{Ad}(N))^{n+1} g_n = 0, (\operatorname{ad}(N))^{n+1} f_n = 0.$$

Essentially, the idea is to take an arbitrary classifying space for Hodge structures, and to consider a part of it as the image of a classifying space given by the upper half plane.

6 Bibliography

1. Schmid, Wilfried, Variation of Hodge Structure: The Singularities of the Period Mapping, Inventiones Math. 22, pp. 211-319 (1973), Springer-Verlag 1973

2. Deligne, Pierre, Travaux de Griffiths, Sem. Bourbaki Exp. 376, Lect. notes in math., 180, Springer (1970) pp. 213235

3. Deligne, Pierre, Théorie de Hodge. I. Actes du Congrs International des Mathmaticiens (Nice, 1970), Tome 1, pp. 425430. Gauthier-Villars, Paris, 1971. MR0441965

4. Deligne, Pierre Théorie de Hodge. II. Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 557.MR0498551

5. Deligne, Pierre Théorie de Hodge. III. Inst. Hautes Études Sci. Publ. Math. No. 44 (1974), 577. MR0498552

6. Deligne, Pierre Structures de Hodge mixtes réelles. Motives (Seattle, WA, 1991), 509514, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994. MR1265541

7. Griffiths, Phillip, Schmid, Wilfried, Recent Developments in Hodge Theory: A Discussion of Techniques and Results, Proceedings of the International Colloquium on Discrete Subgroups of Lie Groups and Applications to Moduli, Bombay, January 1973

8. Griffiths, Phillip, Periods of integrals on algebraic manifolds I (Construction and Properties of the Modular Varieties) Amer. J. Math., 90 (1968) pp. 568626

9. Griffiths, Phillip, Periods of integrals on algebraic manifolds II (Local Study of the Period Mapping) Amer. J. Math., 90 (1968) pp. 808865

10. Griffiths, Phillip, Periods of integrals on algebraic manifolds III. Some global differential-geometric properties of the period mapping. MR0282990 Publ. Math. IHES , 38 (1970) pp. 228296

11. A.I. Ovseevich (2001), “Hodge structure”, in Hazewinkel, Michiel, Encyclopaedia of Mathematics, Kluwer Academic Publishers, ISBN 978-1556080104

12. Saito, Morihiko, Modules de Hodge polarisables, M Saito - Publ. Res. Inst. Math. Sci, 1988

13. Saito, Morihiko, Introduction to mixed Hodge modules. Actes du Colloque de Théorie de Hodge (Luminy, 1987). Astrisque No. 179180 (1989), 10, 145162. MR1042805

14. Saito, Morihiko, Mixed Hodge Modules, Publ. Res. Inst. Math. Sci, 1990

15. Saito, Morihiko, Mixed Hodge Complexes on Algebraic Varieties, Mathematische Annalen, 2000 - Springer

16. J. Steenbrink (2001), "Variation of Hodge structure", in Hazewinkel, Michiel, Encyclopaedia of Mathematics, Kluwer Academic Publishers, ISBN 978-1556080104

17. Steenbrink, Peters, Mixed Hodge Structure, Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume 52, Springer

18. Voisin, Claire, Hodge Theory and Complex Algebraic Geometry I, Cambridge University Press 2002

19. Voisin, Claire, Hodge Theory and Complex Algebraic Geometry II, Cambridge University Press 2002

20. Durfee, Alan, Mixed Hodge structures on punctured neighborhoods, Duke Math. J. Volume 50, Number 4 (1983), 1017-1040.

21. Brocker, Janich, Introduction to Differential Topology, Cambridge University Press, 1982

22. Litt, Daniel, The Poincaré Lemma and de Rham Cohomology, The Harvard College Mathematics Review, Vol. 1, No. 2, 2007.