Étale Fundamental Group: An Exposition Submitted to the Mathematics Department of Harvard College in Partial Fulfillment of the Requirements for the Degree of Artium Baccalaureus

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Chapter 1

Introduction

This paper aims to provide an exposition of the étale fundamental group, which provides a notion of fundamental group for objects called locally Noetherian schemes.

Because we formulate the construction of both schemes and the étale fundamental group in Category-Theoretic language, we begin with an overview of the relevant language and concepts in Category Theory.

Because the étale fundamental group is an analogue of the classical fundamental group defined over a path-connected, semilocally simply connected topological space, we then give a brief review of the construction and properties of the topological fundamental group as a group of homotopy classes of paths. With the notions provided by our Category-Theoretic overview, we are able to give a reformulation of the topological fundamental group suitable for generalization to the context of schemes. This reformulation emphasizes the role of automorphisms of covering spaces. Specifically, it emphasizes the role of automorphisms of the *fiber functor*, the functor associating to each finite covering the preimage of a particular point in the base space.

Theorem 2.2.2: Fix a path-connected, semilocally path-connected, and semilocally simply connected topological space X. The automorphism group $\mathrm{Aut}(\mathcal{F}^{fin}_{X,x})$ of natural transformations from the finite fiber functor to itself is isomorphic to $\widehat{\pi_1^{top}(X,x)}$, the profinite completion of the topological fundamental group of X at the point x.

These automorphisms of covering spaces are analogous to field automorphisms over a base field, the subject of Galois Theory. We therefore provide a brief discussion of Galois Theory, culminating in the construction of the absolute Galois group, the natural analogue of the topological fundamental group, and more precisely, the topological fundamental group under a modification called *profinite completion*, which we will discuss in the section on Category Theory.

Theorem 2.3.8: The Absolute Galois Group $\mathrm{Gal}(^{\Omega}\!/\!_{\mathbb{F}})$ of $\mathbb{F}$ is isomorphic to $\varprojlim Gal(^{\mathbb{L}}\!/\!_{\mathbb{F}})$, the inverse limit of the Galois groups for all Galois extensions $\mathbb{L} \supseteq \mathbb{F}$.

As our object is to define a fundamental group for schemes, we devote the beginning of Chapter 3 to defining schemes and their structure sheaves (Theorem 3.2.1). We establish useful properties for later discussions, including the property of *quasi-compactness*, a compactness notion for non-Hausdorff spaces (Lemma 3.2.2) and the unique extension of a sheaf from basic open sets to general open sets (Lemma 3.2.3).

We then establish a few useful tools for establishing an arbitrary scheme as affine. As many of the properties of étale coverings, the objects of interest to the étale fundamental group, must be established on an affine open cover of the source or target scheme, having tools to generalize properties of affine schemes are valuable to the discussion, and identifying arbitrary schemes as affine is a necessary first step. The properties of the category of affine schemes include closure under disjoint union (Lemma 3.2.4) and finite fiber product (Lemma 3.3.5), and the following results provide several useful properties for later proofs:

Theorem 3.3.2 (Hartshorne Exercise 2.16): Given a quasi-compact scheme $(X, \mathcal{O}_X)$ with a global section f and some affine cover $\{U_\alpha\}$ such that the pairwise intersection $U_\alpha \bigcap U_{\alpha'}$ is quasicompact, the set X_f of points x in X such that the restriction of f to the stalk $\mathcal{O}_{X,x}$ of x is not contained within the maximal ideal $\mathfrak{m}_x$ is an open subscheme of X, and the rings $O_X(X_f)$ and $O_X(X)[\frac{1}{f}]$ are ismorphic.

We then define morphisms of schemes, and develop the ability to determine when certain schemes are isomorphic.

Theorem 3.3.1 (Hartshorne Exercise 2.4): For X, $Spec(A)$ schemes with $Spec(A)$ affine, the mapping $\alpha : Hom_{\mathfrak{Sch}}(X, Spec(A)) \to Hom_{\mathfrak{Ring}}(A, \mathcal{O}_X(X))$ associating to every morphism of schemes $f : X \to Spec(A)$ the induced homomorphism of rings $\varphi_f : A \to \mathcal{O}_X(X)$ is bijective.

Corollary 3.3.4: Let $f : X \to Y$ be a morphism of schemes. Then if there exists an open cover $\{U_\alpha\}$ of Y such that the induced homomorphism of rings $\varphi_\alpha : \mathcal{O}_Y(U_\alpha) \to \mathcal{O}_X(f^{-1}(U_\alpha))$ is an isomorphism for all α, then f is an isomorphism of schemes.

We then restrict our discussion to morphisms which exhibit certain properties, those of being *affine*, *finite*, and *étale*, as morphisms which exhibit all three of these properties form an analogue of *covering* in Topology and *extension* in Galois Theory, and automorphisms of these coverings are used to construct the étale fundamental group. To make these properties easier to work with, we use the last few results above to generalize their properties from specific affine subsets to general affine subsets.

Lemma 3.3.6: Given an affine morphism of schemes $f : X \to Y$ and an open affine subset $U \subseteq Y$, the restriction $f|_{f^{-1}(U)}$ of f to $f^{-1}(U)$ is also affine.

Theorem 3.3.7: A morphism of schemes $f : X \to Y$ is affine if and only if for every open affine U in Y, its preimage $f^{-1}(U)$ is open affine in X.

Theorem 3.3.9: For $f : X \to Y$ a morphism of locally Noetherian affine schemes such that $X \cong Spec(A)$ and $Y \cong Spec(B)$ and f has the property that the induced map of rings $\widehat{f} : B \to A$ takes the form $B \to {}^{B[x]}/_{< h >}$, for h a monic polynomial such that h' is invertible in ${}^{B[x]}/_{< h >}$, then the restriction of f to any distinguished open subset $U_a \to f(U_a)$ has this property as well.

From here, we introduce the natural analogue of finite covering spaces for schemes, étale coverings, and discuss the relevant properties of étale coverings over a fixed space as a category (Theorem 4.1.5), as well as useful properties of objects and morphisms within that category.

Lemma 4.1.7: If $(X \xrightarrow{f} S)$ is a connected object of ${}^{Et}/_{S}$, then any element u of $\mathrm{Hom}_{Et/S}(X, X)$ (the set of morphisms of objects in ${}^{Et}/_{S}$ from X to itself) is an automorphism of X over S.

Lemma 4.1.8: Let (X, x), (Y, y) be a pair of pointed objects in ${}^{Et}\!/\!{}_S$ with X connected. Then if there exists a morphism of pointed objects $u : (X, x) \to (Y, y)$, it is unique.

This discussion allows us to designate particular objects of this category as Galois objects, the natural analogue of Galois field extensions in Galois Theory. We note some interesting and useful properties of these objects, and define a fiber functor for this category analogous to the topological case.

Lemma 4.1.10: An object $(X \xrightarrow{f} S)$ of ${}^{Et}\!/\!{}_S$ is Galois if and only if the fiber product $X \underset{S}{\times} X$ is isomorphic to the disjoint union of a set of copies of X.

Lemma 4.1.11: For $(X \xrightarrow{f_X} S)$, $(Y \xrightarrow{f_Y} S)$, and $(Z \xrightarrow{f_Z} S)$ connected objects of ${}^{Et}\!/\!{}_S$, with Y Galois, then for any two morphisms of objects $g_1, g_2 : X \to Y$, there exists a unique element φ of $Aut({}^Y\!/\!{}_S)$ such that $g_2 = \varphi \circ g_1$, and for any two morphisms of objects $h_1, h_2 : Y \to Z$, there exists a unique element ς of $Aut({}^Y\!/\!{}_S)$ such that $h_2 = h_1 \circ \varsigma$.

We then use these to show that every object is surjected over by the union of finitely many Galois objects and that, in particular, every connected object is surjected over by a unique Galois object, called a *Galois closure*, whose automorphisms completely determine the automorphisms of the objects it surjects over.

Theorem 4.1.12: Any connected object $(Z \xrightarrow{f_Z} S)$ in ${}^{Et}\!/\!{}_S$ has a Galois closure $(X \xrightarrow{f_X} S)$, unique up to isomorphism.

We then define the natural analogue of the fundamental group for schemes, the *étale fundamental group*, as the group of automorphisms of the fiber functor over a point in the base scheme. Our discussion of Galois objects allows us to construct the étale fundamental group out of the automorphism groups of Galois objects.

Theorem 4.2.1: Let $\{P_i\}$ be a collection of Galois objects of ${}^{Et}\!/\!{}_S$ such that for all connected objects X in ${}^{Et}\!/\!{}_S$, there exists some epimorphism $P_i \to X$ for some i (in which case, we say P_i *trivializes* X and $\{P_i\}$ is a *cofinal system* of Galois objects). Then for any s in S, $\pi_1(S, s) \cong \varprojlim_i Aut({}^{P_i}\!/\!{}_S)$.

This construction allows us to demonstrate some useful properties of the étale fundamental group and its action on étale coverings.

Lemma 4.2.3: An object $X \xrightarrow{f_X} S$ of ${}^{Et}\!/\!{}_S$ is connected if and only if $\pi_1(S, s)$ acts on $\mathcal{F}_{Et/S,s}(X)$ transitively.

Lemma 4.2.4: For a connected, nonempty object $X \xrightarrow{f_X} S$ of ${}^{Et}\!/\!{}_S$ and $N \lhd \pi_1(S, s)$ the kernel of the action of $\pi_1(S, s)$ on $\mathcal{F}_{Et/S,s}(X)$, X is Galois if an only if ${}^{\pi_1(S,s)}\!/\!{}_N$ acts freely and transitively on X.

Lemma 4.2.6: For $X \xrightarrow{f_X} S$ and $Y \xrightarrow{f_Y} S$ objects of ${}^{Et}\!/\!{}_S$, morphisms of objects $X \to Y$ bijectively correspond to morphisms of $\pi_1(S, s)-$sets between $\mathcal{F}_{Et/S,s}(X) \to \mathcal{F}_{Et/S,s}(Y)$.

We then compute an example; this example is the scheme associated to a field, in which case, the étale fundamental group is exactly the absolute Galois group of the field. We develop a few tools to help with the construction (Lemmata 4.3.1, 4.3.2, and 4.3.3), and conclude with the following theorem:

Theorem 4.3.4: For $\mathfrak{K}$ a field and k a geometric point of $Spec(\mathfrak{K})$, $\pi_1(Spec(\mathfrak{K}), k)$ is isomorphic to the absolute Galois group of $\mathfrak{K}$.

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1.2 Author's Note

The author is aware that several more concise and rigorous treatments of this subject are widely available to potential students. It is the author's belief, however, that conciseness is often bought at the price of exposition, and terse treatments, however rigorous, are not always useful to new students as learning tools. As this work is intended not only as a demonstration of the author's knowledge but also as a teaching tool, effort has been made to make the subject accessible to students without a thorough grounding in the background fields and to those who have not dealt with this material for some time. The author apologizes if the tone seems redundant or pedantic to the experienced reader, and readers are encouraged to devote their attention to whatever sections they feel are the best use of their time. Effort has been made to keep the tone conversational and explanatory, and while this choice is made at the cost of brevity, it is the author's hope that the finished work is the richer (and the more useful) for it.

Chapter 2

Background

2.1 A Brief Mention of Category Theory

"It is characteristic of the epistemological tradition to present us with partial scenarios and then to demand whole or categorical answers as it were."

-Avrum Stroll

2.1.1 Terminology

Category Theory concerns itself with *Categories*;

Definition 1. A *category* $\mathfrak{C}$ consists of a collection *Ob($\mathfrak{C}$)* of objects of $\mathcal{C}$, equipped with a collection of morphisms *Hom($\mathfrak{C}$)* between these objects. For f an element of Hom($\mathcal{C}$), f: $S \to T$, we say that f is a morphism from S to T, and that S is the *source* and T the *target* of f. We can specify these by saying f is an element of $\mathrm{Hom}(S, T)$.

We also require that there exist an associative composition function of morphisms, including an identity morphism. This is to say, we require that for all R, S, and T in Ob($\mathcal{C}$), there must exist a composition function $\mathrm{Hom}(R, S) \times \mathrm{Hom}(S, T) \to \mathrm{Hom}(R, T)$, such that $(f, g) \mapsto g \circ f$, with $(h \circ g) \circ f = h \circ (g \circ f)$. We also require that for each object S, there exists a unique morphism 1_S in $\mathrm{Hom}(S, S)$ such that for each f in $\mathrm{Hom}(R, S)$ and each $g \in \mathrm{Hom}(S, T)$, $1_S \circ f = f$ and $g \circ 1_S = g$. This 1_S is called the *identity morphism on S*.

2.1.2 Relevant Concepts

A *functor* $\mathcal{F}$ is a mapping of categories which preserves certain structural qualities between the categories.

Definition 2. Given categories $\mathfrak{C}$ and $\mathfrak{D}$, a *functor* $\mathcal{F}$: $\mathfrak{C} \to \mathfrak{D}$ is a mapping which associates to each element S of Ob($\mathfrak{C}$) an element of Ob($\mathfrak{D}$), denoted $\mathcal{F}(S)$ in Ob($\mathfrak{D}$), and to each element f of $\mathrm{Hom}(S, T) \subseteq$ Hom($\mathfrak{C}$) an element, denoted $\mathcal{F}(f)$, of $\mathrm{Hom}(\mathcal{F}(S), \mathcal{F}(T)) \subseteq$ Hom($\mathcal{D}$).

We require of functors two further things: The first is that $\mathcal{F}(1_S) = 1_{\mathcal{F}(S)}$ for all objects S. The second is that either $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all morphisms f and g, in which case $\mathcal{F}$ is called a *covariant functor*, or that $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for all such f and g, in which case $\mathcal{F}$ is called a *contravariant functor*. It should be noted that, unless specifically described as contravariant, functors are assumed to be covariant.

Definition 3. A *natural transformation* is a morphism between covariant functors which preserves structural qualities of the functors themselves. For $\mathcal{F}$ and $\mathcal{G}$, functors from category $\mathfrak{C}$ to category $\mathfrak{D}$, a *natural transformation* ξ from $\mathcal{F}$ to $\mathcal{G}$ is a mapping which associates to every S in $\mathrm{Ob}(\mathfrak{C})$ a morphism ξ_S: $\mathcal{F}(S) \to \mathcal{G}(S)$ such that for every morphism $f : S \to T$ of objects in $\mathrm{Ob}(\mathcal{C})$, $\xi_S \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \xi_T$.

Finally, there are particular objects of a given category $\mathfrak{C}$ which, if they exist, we designate with special distinction.

Definition 4. A *final object* or *terminal object* T in $\mathrm{Ob}(\mathfrak{C})$ of a category $\mathfrak{C}$ is an object for which, for every object X in $\mathrm{Ob}(\mathfrak{C})$, there exists a unique morphism $X \to T$.

Definition 5. An *initial object* T in $\mathrm{Ob}(\mathfrak{C})$ of a category $\mathfrak{C}$ is an object for which, for every object X in $\mathrm{Ob}(\mathfrak{C})$, there exists a unique morphism $I \to X$.

Definition 6. A morphism of objects $f : R \to S$ is called a *monomorphism* if for every pair of morphisms g_1 and g_2 with source some object Q and target R such that the compositions $f \circ g_1$, $f \circ g_2$ are exactly equal, then g_1 and g_2 are exactly equal also. This property is called *left cancellation.*

Definition 7. A morphism of objects $f : R \to S$ is called an *epimorphism* if for every pair of morphisms g_1 and g_2 with source S and target some object T such that the compositions $g_1 \circ f$, $g_2 \circ f$ are exactly equal, then g_1 and g_2 are exactly equal also. This property is called *right cancellation.*

Definition 8. An epimorphism $f : R \to S$ is called *effective* if the fiber product $R \underset{S}{\times} R$ with projection maps π_1, π_2 onto R satisfies the following property: $f \circ \pi_1$ is exactly equal to $f \circ \pi_2$, and for every morphism $g : R \to T$ such that $g \circ \pi_1$ is exactly equal to $g \circ \pi_2$, there exists a unique morphism $g' : S \to T$ such that $g' \circ f$ is exactly g.

Definition 9. A *section* is a right inverse of a morphism. Given a morphism $f : R \to S$, a *section* g of f is a morphism $g : S \to R$ such that $f \circ g$ is the identity on S.

The final two relevant Category-Theoretic concepts are constructions which can be pieced together out of the objects of a category through the equivalence classes imposed by morphisms.

Definition 10. The *pullback* or *fiber product* $R \underset{T}{\times} S$ of two morphisms $f : R \to T$ and $g : S \to T$ is an object equipped with two morphisms $p_1 : R \underset{T}{\times} S \to R$ and $p_2 : R \underset{T}{\times} S \to S$ such that $f \circ p_1 = g \circ p_2$ and such that for any other object Q equipped with morphisms $q_1 : Q \to R$ and $q_2 : Q \to S$ with $f \circ q_1 = g \circ q_2$, there exists a unique morphism $u : Q \to R \underset{T}{\times} S$ such that $q_1 = p_1 \circ u$ and $q_2 = p_2 \circ u$.

We note that this last property makes the fiber product universal.

Definition 11. Let $\mathfrak{T}$ be a functor from some category $\mathfrak{A}$ into $\mathfrak{C}$, and for any object α in $\mathfrak{A}$, let T_α denote the corresponding object in $\mathfrak{C}$. Let the collection $\{T_\alpha\}$ be partially ordered by the existence of morphisms $f_{\alpha,\alpha'} : T_\alpha \to T_{\alpha'}$ such that $f_{\alpha,\alpha'} \circ f_{\alpha',\alpha''} = f_{\alpha,\alpha''}$ and $f_{\alpha,\alpha}$ is the identity map, the *inverse limit* or *projective limit* is the object $\varprojlim_\alpha T_\alpha$ equipped with morphisms $g_\alpha : \varprojlim_\alpha T_\alpha \to T_\alpha$ such that $g_\alpha = f_{\alpha',\alpha} \circ g_{\alpha'}$, and such that every morphism h with $\varprojlim_\alpha T_\alpha$ as its target is equivalent to a set of morphisms $\{h_\alpha\}$ into $\{T_\alpha\}$ which commute with the morphisms $f_{\alpha,\alpha'}$.

Of particular importance in the context of this paper is when these objects are quotient groups of a fixed group G.

Definition 12. For a fixed group G, the *profinite completion* $\widehat{G}$ of G is the inverse limit of groups $\varprojlim_\alpha G/N_\alpha$, where N_α vary over all normal subgroups of G with finite index, and $G/N_\alpha \leqslant G/N_{\alpha'}$ if $N_{\alpha'} \subseteq N_\alpha$.

Finally, there is a Category-Theoretic lemma which we will make use of throughout the course of this paper. Because it applies to any property which is stable under composition and pullback, it is often referred to as the "property p" lemma.

Lemma 2.1.1. *(Property p Lemma): For any property p ascribed to morphisms such that p is stable under composition and pullback, if there exists a commutative diagram*

$$\begin{tikzcd}
Z \arrow[r, "\phi"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \arrow[r, bend left, "id\times id"] & Y \times_X Y \arrow[l] \arrow[d] \\
& X & Y \arrow[l, "g"]
\end{tikzcd}$$

such that the morphisms f and $(id \times id)$ have property p, then ϕ does as well.

Proof. First, we consider the pullback of the maps $(id\times id)$ and (id, ϕ). By inspection, the fiber product is isomorphic to Z, which we illustrate in the following diagrams, where the curved arrow is not a map, but instead represents our filling in the blank spot with Z:

$$\begin{tikzcd}
Y \arrow[r, "id\times id"'] & Y \times_X Y \\
& Y \times_X Z \arrow[u, "(id,\phi)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "id\times id"'] & Y \times_X Y \\
Z \arrow[u, "\phi"'] \arrow[r, "(\phi\times id)"'] & Y \times_X Z \arrow[u, "(id,\phi)"']
\end{tikzcd}$$

Thus, we know $(id \times id)$ exhibits property p and as p is stable under pullback, the map $(\phi \times id)$ is also p.

Next, we examine the following pullback:

$$\begin{tikzcd}
Y \times_X Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "f"] \\
Y \arrow[r, "g"'] & X
\end{tikzcd}$$

We have taken the map f to exhibit property p, and therefore we know that the projection map π_1 also exhibits this property.

Therefore, as we know p to also be stable under composition, the map $\pi_1 \circ (\phi \times id)$ exhibits p. However, this map is exactly ϕ, and so we are done. ■

2.2 The Topological Fundamental Group: The Shape of Things to Come

2.2.1 The Topological Fundamental Group

Definition 13. For X a topological space, a *covering space* over X is a topological space Y equipped with a *covering map* $f : Y \to X$, a continuous map such that for all x in X, there exists an open subset U of X containing x such that $f^{-1}(U) \simeq U \times S$, for S any set equipped with the discrete topology.

Definition 14. *Universal Covering Space*: For a path-connected, semilocally path connected, and semilocally simply-connected topological space X, a *Universal Covering Space* is a path-connected, simply-connected covering space $\widetilde{X} \xrightarrow{\pi} X$ equipped with covering map π.

While covering spaces are in general not unique (in fact, the disjoint union of any number of copies of X can be equipped with the obvious map to form a covering space), for X path-connected, semilocally path connected, and semilocally simply connected, there exists a unique universal covering space $\widetilde{X}$ up to homeomorphism. The proof of this very useful fact is not conceptually difficult, but it is lengthy, and so, for want of space, we defer the curious reader to [Munkres], wherein the construction of the universal covering space is Theorem 82.1.

Theorem 2.2.1. [Homotopy Lifting Principle]: *For $Y \xrightarrow{f} X$ a covering map, p: $[0,1] \to X$ a path in X, $p(0) = x$, and y in the preimage $f^{-1}(x)$ of x, then there exists a unique continuous path $\widetilde{p} : [0,1] \to Y$ such that $f \circ \widetilde{p} = p$ and $\widetilde{p}(0) = y$, called a* lifting *of p, and that for p, p' homotopic in X, $\widetilde{p}$ and $\widetilde{p'}$ are also homotopic in Y, such that the homotopy class of $\widetilde{p}$ depends only on the homotopy class of p.*

Proof. We begin by demonstrating the lifting of a path $p : [0,1] \to X$ from x to x' to a path $\widetilde{p}$ begining at y in $f^{-1}(x)$. We first cover X with open sets $\{U_\alpha\}$ such that the preimage of U_α in Y is homeomorphic to S_α, equipped with the discrete topology. We now subdivide the interval $[0,1]$ into the union of intervals $[s_i, s_{i+1}]$ such that the image of each interval is contained in some U_α. We set $\widetilde{p}(0) = y$, which must be contained by exactly one set $V_\alpha \simeq U_\alpha \times \{s\}$, for s in S. Because the map $f : V_\alpha \to U_\alpha$ is a homeomorphism, we can easily lift p into V_α. Continuing in this way, we can construct $\widetilde{p}$ piecewise through finitely many steps, as the image of p must be compact. As for uniqueness, this follows from the fact that s_n is contained in the n^{th} and $(n+1)^{th}$ U_α involved in these steps, and as the previous step exactly determines $\widetilde{p}(s_n)$, there is only one connected component of $f^{-1}(U_\alpha)$ in which we could place $\widetilde{p}([s_n, s_{n+1}])$ to make $\widetilde{p}$ connected.

We now show that if two paths are homotopic and their lifts begin at the same point, then the liftings are homotopic as well. To do this, we will actually show something stronger, which is that homotopies themselves can be lifted. Suppose $h : [0,1] \times [0,1] \to X$ a homotopy of paths. We first partition $[0,1] \times [0,1]$ into (necessarily finitely many!) compact rectangles $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ such that each rectangle is contained within some U_α. The lifting of paths tells us that $\{0\} \times [0,1]$ and $[0,1] \times \{0\}$ can be lifted appropriately. To fill in the remaining rectangles $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$, we can assume all rectangles $[s_k, s_{k+1}] \times [t_l, t_{l+1}]$ are appropriately lifted for all $k < i$ and all $l < j$. We now note that the previous rectangles uniquely determine the lifting $\widetilde{h}((s_i, t_j))$, and as there is only one connected component of $f^{-1}(U_\alpha)$, with U_α containing $h((s_i, t_j))$, which contains $\widetilde{h}((s_i, t_j))$, and it is homeomorphic to U_α, allowing us to extend $\widetilde{h}$ over $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$. As $[0,1] \times [0,1]$ is compact, we need only repeat this finitely many times, and as above, the construction is unique. Therefore, the liftings of two paths into a covering space which begin at the same point are homotopic if and only if the original paths are homotopic. (The "if" direction follows directly from the continuity of f). ■

Definition 15. *Topological Fundamental Group* The set of homotopy classes of paths in X starting and ending at x form a group under the binary operator concatenation, denoted $\pi_1^{top}(X, x)$, the *Topological Fundamental Group* of X at x.

The construction of this group and proof of its well-definition and properties can be found in [Munkres], wherein they are the subject of section 52.

This group acts on the preimage $f^{-1}(x) \subseteq Y$ by having the homotopy class of p send $\widetilde{p}(0)$ to $\widetilde{p}(1)$, where $\widetilde{p}$ is any lifting of p into Y, for $Y \xrightarrow{f} X$ any covering of X.

Definition 16. The set $f^{-1}(x)$ is called the *fiber over x* in Y.

2.2.2 Finite Covers of Topological Spaces

For the purposes of analogy with finite étale mappings of schemes (to be introduced later), we restrict our discussion of covering spaces to finite covering spaces, which is to say, covering spaces $Y \xrightarrow{f} X$ such that $f^{-1}(x)$ is finite for all x in X.

Definition 17. *Fiber Functor*: It is useful at this point to introduce the *fiber functor*, a functor from the category of topological coverings of a particular space X into $\mathfrak{Set}$, the the category of sets, which associates to each covering $Y \xrightarrow{f} X$ the set $f^{-1}(x)$, the fiber over some fixed x in X, which we denote $\mathcal{F}_{X,x}$.

Definition 18. From this, it is simple to construct the *finite fiber functor* of covering spaces over X, $\mathcal{F}_X^{fin}$, which is the fiber functor restricted to finite covering spaces.

Theorem 2.2.2. *Fix a path-connected, semilocally path-connected, and semilocally simply connected topological space X. The automorphism group $Aut(\mathcal{F}_{X,x}^{fin})$ of natural transformations from the finite fiber functor to itself is isomorphic to $\widehat{\pi_1^{top}(X, x)}$, the profinite completion of the topological fundamental group of X at the point x.*

To clarify, $\mathrm{Aut}(\mathcal{F}^{fin}_{X,x})$ is the group of all sets of mappings $\{\xi_Y : \mathcal{F}^{fin}_{X,x}(Y) \to \mathcal{F}^{fin}_{X,x}(Y)\}$ where $Y \xrightarrow{f} X$ varies over all finite coverings of X, and such that the following diagram commutes for all pointed maps $Y \to Y'$ of finite covering spaces over X:

$$\begin{tikzcd}
\mathcal{F}^{fin}_{X,x}(Y) \arrow[r, "\xi_Y"] \arrow[d] & \mathcal{F}^{fin}_{X,x}(Y) \arrow[d] \\
\mathcal{F}^{fin}_{X,x}(Y') \arrow[r, "\xi_{Y'}"] & \mathcal{F}^{fin}_{X,x}(Y')
\end{tikzcd}$$

Before we prove this Theorem, we must introduce a few tools to help in the proof: First, we introduce the concept of an automorphism of a covering space.

Definition 19. An *automorphism* of a topological covering $Y \xrightarrow{f} X$ is a homeomorphism $\varphi : Y \xrightarrow{\simeq} Y$ such that $f \circ \varphi = f$.

In order to proceed, we would like to be able to apply Lemma 2.1.1, but we must first demonstrate that it is applicable. The following series of lemmata will help us to do so:

Lemma 2.2.3. *Open and closed immersions are stable under composition.*

Proof. Open maps, closed maps, and injective maps are, by inspection, stable under composition. The intersection of these properties must therefore also be. ∎

Lemma 2.2.4. *Open and closed immersions are stable under pullback.*

Proof. We begin by considering the following pullback, wherein f is an open and closed immersion:

$$\begin{tikzcd}
Y \arrow[d, "f"'] & Y \times_Z X \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
Z & X \arrow[l, "g"']
\end{tikzcd}$$

Because f is injective, there is at most one y in the preimage of any point in Z, and so p_2 must also be injective. Because p_2 is a projection, we know it to be an open map as well. Because f is open and closed and g is continuous, we know the set $g^{-1}(f(Y))$ is open and closed in X as well. This subset of X, however, is exactly the image of p_2, and as such, the open map p_2 is bijective onto this subset of X. Thus, the complement in $g^{-1}(f(Y))$ of the open image of the complement of a closed set in $Y \times_Z X$ (which is, by bijectivity onto $g^{-1}(f(Y))$, exactly the image of the closed set) is closed, making p_2 an open and closed immersion. ∎

Lemma 2.2.5. *For $f : Y \to X$ a covering map, the map $(id \times id) : Y \to Y \times_X Y$ is an open and closed immersion.*

Proof. We begin by noting that this diagonal injection is clearly injective. Also, as f is a local homeomorphism, for a small enough open neighborhood U around any point y in Y, the restriction of f to that neighborhood becomes a homeomorphism, and so the preimage of U becomes $S \times U$, where S is the indexing set necessitated by the covering map, and each $\{s\} \times U$ is homeomorphic to U. One of these $\{s\} \times U$ must be the intersection of this set with the diagonal, and as these are disjoint, we know that that set is both the image of U under $(id \times id)$ and homeomorphic to U. Thus, this injection is open.

Now we must show it is closed. We take some covering $\{U_\alpha\}$ of evenly covered neighborhoods of X, and select one of its disjoint copies, which we call $U_{\alpha,\beta}$ in Y. We then take the preimage of one of these $U_{\alpha,\beta}$ under projection in $Y \times_X Y$. Because U_α is an evenly covered neighborhood, the preimage of $U_{\alpha,\beta}$ is homeomorphic to $S \times U_\alpha$. Because these copies are disjoint, we can remove the copy corresponding to the intersection of the preimages of $U_{\alpha,\beta}$ under p_1 and p_2, (or, for the sake of precision, intersecting with the complement of the closure of that copy), and have the remaining set be yet open. We may call this open set $V_{\alpha,\beta}$ in $Y \times_X Y$. From here, we note that the union $\bigcup_{\alpha,\beta} V_{\alpha,\beta}$ must still be open, yet contains every point in $Y \times_X Y$ not along the diagonal, and so the diagonal must be closed.

We therefore have an open, bijective map $(id \times id)$ onto an open and closed subset of $Y \times_X Y$, which makes it necessarily an open and closed immersion. ∎

We note, at the end of this, that we have covered Y with these $U_{\alpha,\beta}$, which are each evenly covered, and that this argument applies for the fiber product of two different covering maps. We therefore conclude the following:

Lemma 2.2.6. *The property of being a covering map is stable under pullback.*

We may now, at long last, demonstrate the following lemma, which will be of great use to us:

Lemma 2.2.7. *Suppose $f : X \to Y$ is a covering map, and $s : Y \to X$ a section of f. Then s is an open and closed immersion.*

Proof. We now have a property, that of being an open and closed immersion, that is stable under pullback and composition, and a diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "id"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

with id and the injection $(id \times id) : Y \to Y \times_X Y$ exhibiting that property. Thus, it follows directly from Lemma 2.1.1 that s is an open and closed immersion. ∎

Lemma 2.2.8. *Given covering maps $f : Y \to Z$ and $g : X \to Z$, any section $s : Y \to Y \times_Z X$ is an open and closed immersion.*

Proof. We know by Lemma 2.2.6 that $Y \times_Z X \to Y$ is a covering map. It then follows from Lemma 2.2.7 that s is an open and closed immersion. ∎

Lemma 2.2.9. *For any two points x and v in X, if there exists a path $q_v : [0,1] \to X$, with $q_v(0) = x$ and $q_v(1) = v$, then $\pi_1^{top}(X,x) \cong \pi_1^{top}(X,v)$. Thus, for a given path component (or for X path-connected), it makes sense to talk about $\pi_1^{top}(X)$.*

Proof. First, fix x and v in X, connected by path $q_v : [0,1] \to X$, with $q_v(0) = x$ and $q_v(1) = v$, and $\widetilde{x}$ in $f^{-1}(x) \subseteq Y$. For g' in $\pi_1^{top}(X,x)$ a homotopy class of loops starting and ending at x, let g be any path representative of g'. Then the concatenation of $q_v \cdot g \cdot q_v^{-1}$ represents a loop beginning and ending at v. Since we can easily make a loop from x out of a loop from v by reversing the conjugation of the concatenation, there is a $1:1$ relationship between homotopy classes of loops at x and v, and so $\pi_1^{top}(X,x) \cong \pi_1^{top}(X,v)$. This also implies that any lifting of the path $q_v \cdot g \cdot q_v^{-1}$ represents a path beginning and ending at points in the fiber over v, the selection of q_v specifies both an isomorphism between the fundamental groups and an action of $\pi_1^{top}(X,x)$ on $f^{-1}(v)$, implying also a homeomorphism between $f^{-1}(x) \simeq f^{-1}(v) \simeq S$, some S with the discrete topology. ∎

Next, we establish a useful property of morphisms of covering spaces.

Lemma 2.2.10. *For $Y \xrightarrow{f_1} X$, $Y' \xrightarrow{f_2} X$ covering spaces of a connected topological space X, with Y connected, if there exists a continuous map $g : Y \to Y'$ such that $f_1 = f_2 \circ g$ bringing y in Y to y' in Y' for any y in Y, it is the only such map to do so.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
Y' \arrow[d, "f_2"] & Y' \underset{X}{\times} Y' \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
X & Y' \arrow[l, "f_2"'] \arrow[u, bend right=60, "id \times id"']
\end{tikzcd}$$

We begin by noting that the composition $p_2 \circ (id \times id)$ is the identity on Y', making $(id \times id)$ a section. From this, we know by Lemma 2.2.8 that $(id \times id)$ is an open and closed immersion. Let us take another map g' from Y' to Y commuting with the covering maps f_1 and f_2 bringing y to y'. We now wish to show $g = g'$.

From here, we consider the pullback $Y \underset{Y \underset{X}{\times} Y}{\times} Y'$ in the following diagram:

$$\begin{tikzcd}
Y \arrow[d, "g \times g'"] & Y \underset{Y \underset{X}{\times} Y}{\times} Y' \arrow[l, "q_1"'] \arrow[d, "q_2"] \\
Y' \underset{X}{\times} Y' & Y' \arrow[l, "id \times id"']
\end{tikzcd}$$

By Lemma 2.2.4, we have shown that as $(id \times id)$ is an open and closed immersion into $Y' \underset{X}{\times} Y'$, q_1 must be as well. As we've taken Y to be connected, this means that the image of q_2 must be either the empty set or all of Y. As $Y \underset{Y \underset{X}{\times} Y}{\times} Y'$, unwinding definitions, amounts to $\{\tilde{y}$ in $Y : g(\tilde{y}) = g'(\tilde{y})\}$, with q_2 either g or g', we already know this set to contain y, and so its image is nonempty. Therefore, the functions g and g' agree on all of Y, and so, $g = g'$. ∎

Lemma 2.2.11. *$\pi_1^{top}(X, x) \cong Aut(\widetilde{X})$, for $\widetilde{X} \xrightarrow{\pi} X$ the universal covering of X.*

Proof. Returning to $\widetilde{X}$, we note that, as it is path-connected, for any two points $\widetilde{x}$ and $\widetilde{x}' \in \pi^{-1}(x)$, there exists a path r connecting the two, and as such, $\pi \circ r$ is a path in X, implying that $\pi_1^{top}(X, x)$ acts transitively on $\pi^{-1}(x)$.
Take now any loop $\widetilde{p} \subseteq \widetilde{X}$ starting from $\widetilde{x}$. As $\widetilde{X}$ is simply connected, $\widetilde{p}$ is contractible to a point through homotopy $h : [0, 1] \times [0, 1] \to \widetilde{X}$ such that $h(0, t) = \widetilde{p}(t)$ and $h(1, t) = h(s, 1) = \widetilde{x}$ for all (s, t) in $[0, 1] \times [0, 1]$. Then $\pi \circ h$ is a homotopy from $\pi \circ \widetilde{p}$ to the constant path x, rendering $\pi \circ \widetilde{p}$ represented by the identity in $\pi_1^{top}(X, x)$, which must therefore act freely on $\pi^{-1}(x)$. Thus, $\pi^{-1}(x)$ is isomorphic to $\pi_1^{top}(X, x)$ as a $\pi_1^{top}(X, x)-$set.

By Lemma 2.2.10, we have the result that it is a universal property of $\widetilde{X}$ that for any $Y \xrightarrow{f} X$, y in $f^{-1}(x)$, $\widetilde{x}$ in $\pi^{-1}(x)$, there exists a unique covering map $\widetilde{X} \xrightarrow{g} Y$ such that g: $\widetilde{x} \mapsto y$ and $f \circ g = pi$. From here, we can surmise that, as $\widetilde{X}$ is a covering space of X, for any two $\widetilde{x}$, $\widetilde{x}'$ in $\pi^{-1}(x)$ there exists a unique covering map g': $\widetilde{X} \to \widetilde{X}$ such that $\widetilde{x} \mapsto \widetilde{x}'$ and $\pi \circ g' = \pi$. As g' is a covering map, it is a local homeomorphism surjective over $\widetilde{X}$, and, invoking Lemma 2.2.10 again, invertible, which makes it a bijective local homeomorphism. Thus, it is a homeomorphism, which makes it an automorphism of $\widetilde{X} \xrightarrow{\pi} X$. Note also that any such automorphism is also a covering map bringing elements of the fiber to one another, and that there is therefore a unique automorphism bringing any given $\widetilde{x}$ to a given $\widetilde{x}'$. Therefore, $\mathrm{Aut}(\widetilde{X})$ acts freely and transitively on $\pi^{-1}(x)$, as does $\pi_1^{top}(X, x)$, rendering them isomorphic as groups. ∎

This reduces our proof of Theorem 2.2.2 to the following: Show $\mathrm{Aut}(\mathcal{F}_X^{fin}) \cong$ $\widehat{Aut(\widetilde{X})}$. For ease of notation, let us denote the group $Aut(\widetilde{X})$ as G.

Proof. (Theorem 2.2.2) Now, we take any $Y \xrightarrow{f} X$, and equip $\tilde{X}$ with covering maps g_1, g_2: $\tilde{X} \to Y$ and Y with automorphism ν such that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "g_1"] \arrow[dr, "g_2"] \arrow[drr, bend right=60, "\pi"'] & Y \arrow[d, "\nu"] \arrow[dr, "f"] & \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

The specification of g_1 and ν uniquely determine g_2 as the unique pointed mapping bringing $\tilde{x} \mapsto \nu \circ g_1(\tilde{x})$. However, there exists some $\tilde{x}'$ in $g_1^{-1}(g_2(\tilde{x})) \subseteq \pi^{-1}(x)$, and so there must exist automorphism χ: $\tilde{X} \to \tilde{X}$, $\tilde{x} \mapsto \tilde{x}'$. Therefore, every automorphism of a covering space over X is determined by a (not generally unique) automorphism of $\tilde{X}$:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "g_1"] \arrow[d, "\chi"] \arrow[dr, "g_2"] & Y \arrow[d, "\nu"] \arrow[dr, "f"] & \\ \widetilde{X} \arrow[r, "g_1"] & Y \arrow[r, "f"] & X \end{tikzcd}$$

Take $Y \xrightarrow{f} X$, with $G \circlearrowright f^{-1}(x) \subseteq Y$ in the way specified above. Because any G-set is the disjoint union of its orbits, we can assume Y connected such that $G \circlearrowright f^{-1}(x)$ transitively without loss of generality, since $\mathrm{Aut}(Y \coprod Y')$ is determined by $\mathrm{Aut}(Y)$ and $\mathrm{Aut}(Y')$. This, as above, guarantees that G acts transitively on the fibers of Y over X.

We recall from Group Theory that every transitive G-action on a set (call it Z) is isomorphic to its action on left H-cosets by left-multiplication for some subgroup $H \subseteq G$, the stabilizer of any z in Z. Also, $h \cdot H \cdot h^{-1} = \mathrm{Stab}(h \cdot z)$ for all h in G.

For the time being, we restrict our discussion to the case in which this $H \triangleleft G$ is normal, which, to associate it with a particular Y, we will denote $N_Y \triangleleft G$. Now, $f^{-1} \cong {}^{G}/_{N_Y}$ as G-sets, and Lemma 2.2.10 above tells us that $\mathrm{Aut}(Y \to X)$ acts freely and transitively on $f^{-1}(x)$ as ${}^{G}/_{N_Y}$ does on ${}^{G}/_{N_Y}$. Thus, $\mathrm{Aut}(Y \to X) \cong {}^{G}/_{N_Y}$ as groups.

Definition 20. For Y a topological space and $\sim$ an equivalence relation on Y, we can create a *quotient space* ${}^{Y}/_{\sim}$ whose points are the equivalence classes of points of Y under $\sim$. We topologize this space with the *quotient topology*, which has as open sets those sets with open preimages under the map $Y \to {}^{Y}/_{\sim}$, which sends each point y in Y to its equivalence class under $\sim$. Points y, y' in Y such that $y \sim y'$ are said to be *glued together* under this map.

Definition 21. (Galois Covering) Note also that for any normal $N \triangleleft G$, we can create a quotient map gluing $N \cdot \tilde{x}$ (each N-orbit) together creating a quotient space and covering map ${}^{\tilde{X}}/_{N} \longrightarrow X$. We then denote ${}^{\tilde{X}}/_{N}$ as Y^N.

For N of finite index in G, Y^N is then a finite cover of X, and all finite coverings $Y \xrightarrow{f_Y} X$ with automorphism groups finite quotient groups of G can be created in this way (or are isomorphic to one created this way). Such a covering is called a *normal* or *Galois* covering.

Now, let the following be a pointed map of finite covering spaces.

$$\begin{tikzcd} Y \arrow[r, "q_{Y,Y'}"] \arrow[dr, "f_Y"'] & Y' \arrow[d, "f_{Y'}"] \\ & X \end{tikzcd}$$

$q_{Y,Y'}$ induces a surjective homomorphism $\widetilde{q}_{Y,Y'} : \mathrm{Aut}(Y \to X) \twoheadrightarrow \mathrm{Aut}(Y' \to X)$, where $\widetilde{q}_{Y,Y'} : \varphi_Y \mapsto q_{Y,Y'} \circ \varphi \circ q^{-1}_{Y,Y'}$. Now, $\widetilde{q}_{Y,Y'} : \varphi_Y \mapsto q_{Y,Y'} \circ \varphi \circ q^{-1}_{Y,Y'}$ is given both its well-definition and surjectivity from the unique existence of such a map, with the added note that $\mathrm{Aut}(Y' \to X) \circlearrowright f^{-1}_{Y'}(x)$ freely and transitively, means that a mapping whose image acts transitively is therefore necessarily surjective.

From here, $\mathrm{Aut}(Y' \to X) \cong {}^{Aut(Y \to X)}/_{N_{Y,Y'}}$, for some normal $N_{Y,Y'} \lhd \mathrm{Aut}(Y \to X)$. This means that $\mathrm{Aut}(Y' \to X)$ is contained within $\mathrm{Aut}(Y \to X)$ as a subgroup.

We now create a partial ordering of finite covering spaces, ordered by the existence of such a map (i.e. $Y \geqslant Y'$ if such a $q_{Y,Y'}$ exists). Note also that this partially ordered set is identical to that created by partially ordering their automorphism groups by inclusion.

As an aside, we recall from group theory that any subgroup of finite index contains a normal subgroup of finite index, and therefore all finite covering spaces $Y' \leqslant Y$ for Y some finite Galois covering. The existence of this surjective map means that any set of finite covering space-morphisms $\{\varphi_Y | Y \to X \text{ finite }\}$ which commute with pointed maps $\varphi_{Y'}$ entirely determined by φ_Y. We therefore may reduce $\mathrm{Aut}(\mathcal{F}^{fin})$ to the set of finite covering-space automorphisms
$\{\varphi_Y | Y \to X \text{ a finite Galois covering}\}$ which commute with pointed maps. Fortunately, as these maps induce a partial ordering on the automorphism groups connected by surjective homomorphism, we can create $\mathrm{Aut}(\mathcal{F}^{fin}) \cong \varprojlim Aut(Y \to X)$ for $Y \to X$ normal.
$\cong \varprojlim {}^{G}/_{N_Y}$ for $Y \to X$ normal.
$\cong \varprojlim {}^{G}/_{N}$ for $N \lhd G$ normal.
$\cong \widehat{\pi_1^{top}(X, x)}$. ■

2.3 Galois Theory: Further Afield

If the above correspondence between subgroups of an automorphism group and surjectively-mapping-space sounds disconcertingly familiar to previous students of Galois Theory, such students are in excellent company. In fact, it is partially by deep result (as we will see) and partially by design (restriction to finite covering spaces) that the above example so closely mirrors the fundamental results of Galois Theory. For those less familiar, we provide the following primer, in which we must quote all relevant information directly from Chapter 7 of [Cox] without proof for want of space.

Definition 22. An *ideal* of a ring $\mathcal{R}$ is a subset $I \subseteq \mathcal{R}$ such that, for any i, i' in I and any r, r' in $\mathcal{R}$, the element $(i \cdot r) + (r' \cdot i')$ is also in I, for $+$ and $\cdot$ the additive and multiplicative binary operations on $\mathcal{R}$ respectively

Definition 23. The uniquely smallest ideal of a commutative ring $\mathcal{R}$ which contains an element r is called the ideal *generated by* r, and is denoted $< r >$.

Definition 24. For $\mathfrak{I}$ any ideal of a commutative ring with unit $\mathcal{R}$, we define a *quotient map* to be a mapping φ from $\mathcal{R}$ to the set of equivalence classes $\{r + \mathfrak{I}\}_{r \in \mathcal{R}}$ (such that any two elements r, s in $\mathcal{R}$ are in the same equivalence class if there exists some i in $\mathfrak{I}$ such that $r + i = s$) which maps an element r to its equivalence class. By inspection, this set inherits from $\mathcal{R}$ the structure of a commutative ring with unit, which we denote the *quotient ring* ${}^{\mathcal{R}}/_{\mathfrak{I}}$, and which makes φ a homomorphism. This ring is isomorphic to the target of any surjective homomorphism of rings $\varphi' : \mathcal{R} \to \mathcal{S}$ such that the kernel $\varphi'^{-1}(0)$ is exactly $\mathfrak{p}$.

Definition 25. An ideal I of a commutative ring $\mathcal{R}$ is called *prime* if, whenever elements a and b of $\mathcal{R}$ satisfy $a \cdot b$ an element of I, then a or b or both are contained in I as well. A prime ideal is called *maximal* if it is the only proper ideal which contains all its elements.

Definition 26. A commutative ring with unit $\mathcal{R}$ is called an *integral domain* if the ideal $\{0\}$ in $\mathcal{R}$ is a prime ideal. It is called a *field* if every element which is not the additive identity has a multiplicative inverse.

Lemma 2.3.1. *The quotient ring* $\mathcal{R}/\mathfrak{I}$ *is an integral domain if and only if* $\mathfrak{I}$ *is prime in* $\mathcal{R}$*, and it is a field if and only if* $\mathfrak{I}$ *is maximal in* $\mathcal{R}$*.*

For $\mathcal{R}$ a commutative ring with unit, it is often useful to addend elements with specific properties through ring adjunction. The simplest adjoined element is a formal variable which interacts with the other elements of $\mathcal{R}$ only as determined by the formal binary operators without any special relations. However, to instill useful properties into the variables it is often necessary to force relations by adjoining additional elements specifically to act in these relations in quotient rings. For example, if $\mathcal{R}[x]$ requires that x have a multiplicative inverse, the quotient ring $\mathcal{R}[x][y]/<x \cdot y - 1>$ associates the ideal generated by $x \cdot y - 1$ to the additive identity, rendering y the appropriate inverse to x. More generally, we can adjoin an element α to the ring $\mathcal{R}$ through the *evaluation homomorphism* $\mathcal{R}[x] \to \mathcal{R}[\alpha] = \{f(\alpha) | f(x) \in \mathcal{R}[x]\}$.

Adjunction is also used in fields. For $\mathbb{F}$ a field, it may be necessary to add elements with various properties, depending on our purposes, often the roots of polynomial equations. For example it may be particularly useful for an element a in $\mathbb{F}$ to have a square root, where currently it does not. In this case, the quotient ring $\mathbb{F}[x]/<x^2 - a>$ will provide a square root to a, with $x \cdot a^{-1}$ its inverse, but in this case, either the image of x or that of its additive inverse can be used as a square root of a.

Adding such an element creates a new field entirely, which we will call $\mathbb{L}$. Such a field can be considered a vector space over $\mathbb{F}$, wherein $\mathbb{L} \cong \mathbb{F} \times \alpha \cdot \mathbb{F} \times \alpha^2 \cdot \mathbb{F} \times ...$, with the differing powers of α forming a basis over $\mathbb{F}$.

Definition 27. A field $\mathbb{L}$ is called an *extension over* $\mathbb{F}$ if there exists an injective homomorphism of fields $\mathbb{F} \to \mathbb{L}$. In this case, we identify $\mathbb{F}$ with its image under this homomorphism, and may refer to $\mathbb{F} \subseteq \mathbb{L}$ as a *subfield* of $\mathbb{L}$

Of course, if α is the root of a polynomial equation over $\mathbb{F}$, it satisfies a relation that will render only finitely many of these dimensions linearly independent.

Definition 28. An element α of $\mathbb{F}$ is called *algebraic over* $\mathbb{F}$ if there exists some polynomial f in $\mathbb{F}[\alpha][x]$ such that all coefficients of f are in the image of the inclusion $\mathbb{F} \to \mathbb{F}[\alpha][x]$ and f maps to the additive identity under the evaluation morphism $\mathbb{F}[\alpha][x]/<x - \alpha>$. Intuitively, we can consider this equivalent to saying that α is the root of a polynomial f' in $\mathbb{F}[x]$. If α is not *algebraic* over $\mathbb{F}$, we say it is *transcendental.*

Lemma 2.3.2. *For* $\mathbb{F}$ *a field, and* α *algebraic over* $\mathbb{F}$*, the ring* $\mathbb{F}[\alpha]$ *is again a field.*

As such, we get the following:

Lemma 2.3.3. [Primitive Element Theorem]: *For $\mathbb{L}$ a separable extension of $\mathbb{F}$, $\mathbb{L}$ is a finite-dimensional $\mathbb{F}$-vector space if and only if it is isomorphic to $\mathbb{F}[\alpha]$ for some α algebraic over $\mathbb{F}$.*

Definition 29. In this case, we call the dimension the *degree* of $\mathbb{L}$ over $\mathbb{F}$, denoted $[\mathbb{L} : \mathbb{F}]$. If α is not algebraic over $\mathbb{F}$, we say $[\mathbb{F}(\alpha) : \mathbb{F}] = \infty$.

Definition 30. An extension $\mathbb{L} \supseteq \mathbb{F}$ is called *algebraic* if every element in $\mathbb{L}$ is algebraic over $\mathbb{F}$.

Definition 31. Similarly, an extension $\mathbb{L}$ over $\mathbb{F}$ is *separable* if, for all α in $\mathbb{L}$, the minimal polynomial of α over $\mathbb{F}$ is *separable*, or has distinct roots (which is to say, it is square-free when split into linear factors).

Take note that we will restrict our discussion to separable extensions in the interest of scope: All extensions and polynomials may be assumed to be separable from this point onward.

Such an α is generally a root of several such polynomials with coefficients in $\mathbb{F}$, but there is one of particular importance.

Definition 32. The *minimal polynomial* of α over $\mathbb{F}$ is the unique monic polynomial f such that for all polynomials g in $\mathbb{F}[x]$ with $f(\alpha) = 0$, g is a multiple of f.

Lemma 2.3.4. *For α algebraic over $\mathbb{F}$, $f(\alpha) = 0$ and f irreducible in $\mathbb{F}$ if and only if f is the minimal polynomial of α over $\mathbb{F}$.*

Now, any field extension $\mathbb{K}$ over $\mathbb{L}$ is automatically a field extension over $\mathbb{F}$. The degree $[\mathbb{K} : \mathbb{F}] = [\mathbb{K} : \mathbb{L}] \cdot [\mathbb{L} : \mathbb{F}]$. With this transitivity, we can construct a partial ordering of all fields by inclusion, where $\mathbb{F} \leqslant \mathbb{K}$ if there exists an injective homomorphism of fields $\mathbb{F} \to \mathbb{K}$. The chains formed by this arrangement are often referred to as *towers* of fields.

Of particular importance are extensions called *splitting fields.*

Definition 33. The *splitting field* of a monic, non-constant polynomial f in $\mathbb{F}[x]$ is the smallest field $\mathbb{L}$ containing $\mathbb{F}$ such that f factors (or "*splits*") into linear factors $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$, α_i in $\mathbb{L}$. This field is $\mathbb{L} = \mathbb{F}(\alpha_1, ..., \alpha_n)$, and it is unique up to a non-unique isomorphism to any other splitting field of f over $\mathbb{F}$ which carries the image of $\mathbb{F}$ from one injection to its image in the other.

Definition 34. Such an isomorphism $\mathbb{L} \to \mathbb{L}$ for $\mathbb{L}$ an extension of $\mathbb{F}$, which preserves the image of $\mathbb{F}$ in $\mathbb{L}$ is called an *automorphism* of $\mathbb{L}$ over $\mathbb{F}$, or an $\mathbb{F}$-*automorphism* of $\mathbb{L}$.

Similarly, for, α_i, α_j zeroes of the same irreducible separable polynomial in $\mathbb{F}[x]$, there exists an isomorphism $\mathbb{F}(\alpha_i) \xrightarrow{\cong} \mathbb{F}(\alpha_j)$ which preserves $\mathbb{F}$. This isomorphism can be extended to an automorphism of the splitting field which carries $\mathbb{F}(\alpha_i)$ to $\mathbb{F}(\alpha_j)$ while preserving $\mathbb{F}$ underneath. Not all field extensions form the splitting field of any polynomial. In fact:

Lemma 2.3.5. *For $\mathbb{L}$ the splitting field of f in $\mathbb{F}[x]$, g in $\mathbb{F}[x]$ irreducible, g either splits completely in $\mathbb{L}$ or is irreducible in $\mathbb{L}[x]$ as well.*

This leads to the concept of a *normal* extension.

Definition 35. A *normal extension* is an extension $\mathbb{L} \supseteq \mathbb{F}$ such that every irreducible g in $\mathbb{F}[x]$ either splits completely or is irreducible in $\mathbb{L}$.

All splitting fields are normal extensions, and all normal extensions of finite degree are splitting fields. As our focus is algebraic extensions, we may use the terms interchangeably in the context of finite degree. The term *Galois Extension* may also be used to refer to finite normal field extensions.

Definition 36. The adjunction of one or more roots of an irreducible polynomial which do not generate all conjugate roots creates a field extension which is not normal. Such an extension is called an *intermediate field* $\mathbb{K}$ between the base field $\mathbb{F}$ and the splitting field $\mathbb{L}$, such that $\mathbb{L} \supseteq \mathbb{K} \supseteq \mathbb{F}$ is a tower of fields.

Splitting fields are also normal extensions over their intermediate fields, and just as there exists a group of field automorphisms of $\mathbb{L}$ which fix $\mathbb{F}$, a subgroup of these automorphisms fix $\mathbb{K}$.

Definition 37. The group of automorphisms of a field extension which fixes the base field is called the *Galois Group G* of the extension, or *Gal($\mathbb{L}/\mathbb{F}$)*. These automorphisms act by permuting the conjugate roots of the polynomial associated to the splitting field.

For $\{\alpha_i\}$ in $\mathbb{K}$, only those elements of G which fix $\{\alpha_i\}$ are elements of $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$.

Theorem 2.3.6. *(The Fundamental Theorem of Galois Theory) For $\mathbb{L} \supseteq \mathbb{F}$ a Galois extension, intermediate fields exist in bijective correspondence to subgroups of Gal($\mathbb{L}/\mathbb{F}$), with an intermediate field $\mathbb{K}$ corresponding to its stabilizer under the action of Gal($\mathbb{L}/\mathbb{F}$) on its elements. This correspondence associates to each subgroup the largest intermediate field fixed by the action of Gal($\mathbb{L}/\mathbb{F}$) on the elements of $\mathbb{L}$ by permuting conjugate roots, called its* fixed field.

As the permutation of these roots generates field automorphisms, it should come as no surprise that they are, in many ways, algebraically interchangeable up to the action of the Galois group, and in fact, the fixed fields of conjugate subgroups are isomorphic to one another, as all conjugate roots satisfy the same minimal relation required for them to interact with elements of $\mathbb{F}$ in any meaningful way.

Theorem 2.3.7. *The fixed field of a normal subgroup of the galois group of a normal extension is itself a normal extension over the base field.*

Proof. For $\mathbb{L} \supseteq \mathbb{K} \supseteq \mathbb{F}$, $\mathbb{L}$ normal over $\mathbb{F}$, and ς in $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$, we call $\varsigma\mathbb{K}$ its conjugate field, and as group theory dictates, the stabilizer of $\varsigma\mathbb{K}$ is $\varsigma H \varsigma^{-1}$ for H the stabilizer of $\mathbb{K}$. A normal subgroup, unique in its conjugacy class, is associated to a field such that conjugation of the roots does not affect the field. Thus, for g in $\mathrm{Gal}(\mathbb{L}/\mathbb{F})$, and

α in $\mathbb{K}^H$ the fixed field of H a root of separable f in $\mathbb{F}$, $g \cdot \alpha$ in $\mathbb{K}^H$ as well. Now, as the Galois group of a normal extension acts transitively on the set of conjugate roots of a particular irreducible f, this means that $\mathbb{K}$ consists of the union of whole Gal($\mathbb{L}/\mathbb{F}$)-orbits, and so either an irreducible polynomial in $\mathbb{F}$ splits completely in $\mathbb{K}$ or remains irreducible, so $\mathbb{K} \supseteq \mathbb{F}$ must be a normal extension as well. ■

We can then introduce $\Omega \supseteq \mathbb{F}$ the *separable closure* of $\mathbb{F}$.

Definition 38. For $\mathbb{F}$ a field, the *separable closure* Ω of $\mathbb{F}$ the unique (up to isomorphism) field containing $\mathbb{F}$ in which all separable elements of $\mathbb{F}[x]$ split completely but such that every element α in Ω is algebraic over $\mathbb{F}$.

While this is clearly and by construction a normal extension, note that it is not generally Galois, as the extension is not generally finite. However, we can still describe the group Gal($\Omega/\mathbb{F}$) of $\mathbb{F}$-preserving automorphisms of Ω.

Definition 39. The group Gal($\Omega/\mathbb{F}$) of $\mathbb{F}$-preserving automorphisms of Ω is called the *Absolute Galois Group* of $\mathbb{F}$.

Theorem 2.3.8. *The Absolute Galois Group $Gal(^{\Omega}/_{\mathbb{F}})$ of $\mathbb{F}$ is isomorphic to $\varprojlim Gal(^{\mathbb{L}}/_{\mathbb{F}})$ for all Galois extensions $\mathbb{L} \supseteq \mathbb{F}$.*

Proof. In fact, we can recover the action of the Galois groups of all intermediate fields on conjugate roots directly from the action of Gal($\Omega/\mathbb{F}$). The uniqueness (up to isomorphism) of a splitting field means that Ω must also contain as subfields all Galois extensions $\mathbb{L}$ of $\mathbb{F}$, and must therefore also have a group of $\mathbb{L}$-preserving automorphisms. Any $\mathbb{F}$-preserving automorphism over $\mathbb{L}$ can be extended into an automorphism of Ω, and so there must exist a surjective homomorphism π: Gal($\Omega/\mathbb{F}$)$\twoheadrightarrow$Gal($\mathbb{L}/\mathbb{F}$). The latter is finite, and so the kernel of this surjection must be a normal subgroup of finite index. And because the Galois Group of a given Galois extension determines the behavior of its intermediate fields, we need only consider the Galois groups of Galois extensions in determining the equivalence of the Absolute Galois Group and the projective limit of the Galois Groups of Galois extensions. ■

It will not have escaped the reader's attention that we can consider the Absolute Galois Group's governance of the behavior of Galois Groups of finite extensions as analogous to that of $\widehat{\pi_1^{top}(X,x)}$ on automorphisms of finite covering spaces of X, with Galois extensions corresponding to Galois coverings. In some sense (which we will make rigorous later) we are able to construct out of the conjugate roots $\{\alpha_i | f(\alpha_i) = 0\}$ a fiber over the image of x in the composed mapping $\mathbb{F}[x] \rightarrow {}^{\mathbb{F}[x]}/_{<f>} \rightarrow \Omega$. Such a construction, however, relies on the tools provided by objects known as *schemes*, which we discuss next.

Chapter 3

Schemes and Sheaves

This analogy was formalized by Alexander Grothendieck, who discovered working with a class of mathematical object called *schemes*, which are of use in generalizing the algebraic varieties of rings, that the notions of finite field extension and finite topological coverings could both be generalized in the language of scheme morphisms. We devote this chapter to a discussion of the structure of these objects. However, the structure of a scheme is provided by an overlaid object called a *sheaf*, which merits a small digression.

3.1 Sheaves

"For life is tendency, and the essence of a tendency is to develop in the form of a sheaf, creating, by its very growth, divergent directions among which its impetus is divided."

-Henri Bergson

Rigorously speaking, a *sheaf* is a *presheaf* which satisfies certain special conditions, and so we will begin by defining the *presheaf*.

Definition 40. A *presheaf* over a topological space X is a contravariant functor from the category $\mathfrak{Open}(X)$ of open sets of X (whose morphisms are provided by inclusion maps) to another category $\mathfrak{C}$. For our purposes, we will be discussing only the case in which $\mathfrak{C}$ is the category $\mathfrak{Ring}$ of commutative rings with unit. A *presheaf of commutative rings with unit* $\mathcal{O}$ is a mapping which associates to each open set U of a topological space X a commutative ring with unit $\mathcal{O}(U)$, and to each inclusion of open sets $V \subseteq U \subseteq X$ a homomorphism of rings $res_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$ which obeys the following properties:

1. $res_{U,U}$ is the identity map on $\mathcal{O}(U)$ for all open subsets $U \subseteq X$.

2. The restriction maps must commute: for all open sets U, V, W in X, $W \subseteq V \subseteq W \subseteq X$, $res_{U,W} = res_{V,W} \circ res_{U,V}$. Note that the order of composition is what gives contravariance.

For such a presheaf to qualify as a *sheaf of commutative rings*, it must also satisfy two properties known as the *sheaf axioms*

Definition 41. A *sheaf* is a presheaf which satisfies the following sheaf axioms:

1. The Local Identity Axiom: For any $\{U_i\}$ such that $\bigcup_i U_i = U$ is an open cover of $U \subseteq X$ open, then for any s, t in $\mathcal{O}(U)$ such that $res_{U,U_i}(s) = res_{U,U_i}(t)$ for all i, then $s = t$.

2. The Gluing Axiom: For any $\{U_i\}$ such that $\bigcup_i U_i = U$ is an open cover of $U \subseteq X$ open, then for every set $\{s_i : s_i \text{ in } \mathcal{O}(U_i)\}_i$ such that $res_{U_i, U_i \bigcap U_j}(s_i) = res_{U_j, U_j \bigcap U_i}(s_j)$, then there exists s in $\mathcal{O}(U)$ such that $res_{U,U_i}(s) = s_i$ for all i.

These are sometimes combined for the sake of elegance into a single axiom, which states that for any $\{U_i\}$, $\bigcup_i U_i = U$ an open cover of $U \subseteq X$ open, then the ordered set of mappings $(res_{U,U_i}) : \mathcal{O}(U) \to \prod\limits_i \mathcal{O}(U_i)$ is an injective map whose image consists of those families $\{s_i : s_i \text{ in } \mathcal{O}(U_i)\}$ whose restriction morphisms agree pairwise on the intersection of any two elements of the cover. This is to say, for every such family, there exists a unique element s in $\mathcal{O}(U)$ such that $res_{U,U_i}(s) = s_i$ for all i. (The section guaranteed by the Gluing Axiom is unique). Often, this axiom is glibly summarized in the following way:

Lemma 3.1.1. *A presheaf of commutative rings $\mathcal{O}$ is a sheaf if and only if the following sequence is exact for every open set U of X and every covering $\{U_i\}$ of U:*
$$0 \to \mathcal{O}(U) \to \prod_i \mathcal{O}(U_i) \rightrightarrows \prod_{i,j} U_i \bigcap U_j \to 0,$$
where the first arrow represents the only homomorphism from the trivial ring, the second arrow represents the mapping (res_{U,U_i}), and the pair of arrows together has as its kernel the difference kernel of the pair of mappings $res_{U_i,U_i \bigcap U_j}$ and $res_{U_j,U_i \bigcap U_j}$.

Definition 42. The *difference kernel* or *binary equaliser* of two morphisms $f, g : X \to Y$ consists of all points x in X such that $f(x) = g(x)$ in Y. It can be thought of as the kernel of the map $x \mapsto (f - g)(x)$, or, in the language of fiber products, the intersection $X \underset{Y}{\times} X \bigcap \{(x, x) \in X \times X\}$ of the fiber product of f and g with the diagonal of $X \times X$.

For our purposes, it is salient only that the kernel of the double-arrow mapping (and, by exactness, the image of the injective mapping $\mathcal{O}(U) \to \prod\limits_i \mathcal{O}(U_i)$) consists exactly of those elements $(s_i) \in \prod\limits_i \mathcal{O}(U_i) \mapsto 0$ such that $res_{U_i,U_i \bigcap U_j}(s_i) = res_{U_j,U_i \bigcap U_j}(s_j)$.

Definition 43. The elements of the ring $\mathcal{O}(U)$ are called the *sections* of $\mathcal{O}$ over U. The sections of $\mathcal{O}(X)$ are called *global sections*.

If these three equivalent definitions of a sheaf seem redundant, this is intentional. Sheaves are a difficult topic upon first approach, and often a differing initial perspective aides in understanding. If the concept is still difficult, it may help to consider the metaphor of the sheaf itself. The idea is that each element of $\mathcal{O}(X)$, the *global sections*, represents the end of a stem of grain, the length of which weaves through each of the contained open sets, assuming a slightly different shape at each point along the way. Each stem winds differently, but at each point along its length, those nearby are (at the risk of punning) bundled together by a ring, not unlike a sheaf of grain.

It is also often useful to discuss the behavior of a sheaf at a point x. Inconveniently, sheaves do not associate rings to points, only to open sets, and $\{x\}$ is rarely an open set. We might instead consider looking at the behavior of the sheaf on the smallest open set containing x, but again, under most topologies, such a thing does not usually exist. Taking the intersection of all open sets which contain x would get us closer, but with no guarantee that the resulting set would be open with a ring associated to it. The solution to this problem is a vague analogy of the above attempts, but done over the rings associated to the open sets rather than the sets themselves.

Definition 44. For a sheaf $\mathcal{O}$ and a point x, we call the *Stalk of* $\mathcal{O}$ *over* x the direct limit of the rings $\mathcal{O}(U)$ for all open sets U containing x;
This we denote $\mathcal{O}_x := \varinjlim_{U \ni x} \mathcal{O}(U) = {}^{\amalg_{U \ni x} \mathcal{O}(U)}/_{\sim}$, where for u in U and v in V, with U and V open sets of X, $u \sim v$ if there exists some open set $W \subseteq U \bigcap V$ with W containing x, such that $res_{U,W}(u) = res_{V,W}(v)$.

3.2 Schemes

"The mind is never satisfied with the objects immediately before it, but is always breaking away from the present moment, and losing itself in schemes of future felicity."

-Samuel Johnson

3.2.1 The Affine Case: The Best-Laid Schemes

A sheaf over any category can be laid over any topological space, but Grothendieck's insight was to overlay a sheaf of rings onto a ring itself, or rather, onto a ring's spectrum topologized under the Zariski topology.

Definition 45. The *spectrum*, $Spec(\mathcal{R})$, of a commutative ring with unit $\mathcal{R}$ is the set of prime ideals I in $\mathcal{R}$.

The *Zariski Topology* topologizes this set with a basis of open sets each associated as the *distinguished open set* U_f of a particular element f of $\mathcal{R}$. We will define this rigorously momentarily, but in order to understand these basic open sets, we must first specify a method of turning an element f of $\mathcal{R}$ into a quasi-function over $Spec(\mathcal{R})$

(we say "quasi-function" because it sends different elements of $Spec(\mathcal{R})$ to different targets).

For a given p in $Spec(\mathcal{R})$ corresponding to the prime ideal $\mathfrak{p}$ in $\mathcal{R}$, the fact that $\mathfrak{p}$ is prime in $\mathcal{R}$ guarantees that the quotient ring $\mathcal{R}/\mathfrak{p}$ is an integral domain, to which we can then adjoin multiplicative inverses to all non-units to form $\mathcal{K}(p)$, the *quotient field* or *field of fractions* of the ring $\mathcal{R}/\mathfrak{p}$. It is this field into which we define the quasifunction associated to an element f of $\mathcal{R}$. At the risk of abusing notation, we say $f : Spec(\mathcal{R}) \to \mathcal{K}(p)$, where $f : p \mapsto (\phi \circ \pi)(f)$, for π the quotient map $\mathcal{R} \to \mathcal{R}/\mathfrak{p}$ associating f to the equivalence class $\{f + q | q \in \mathfrak{p}\}$ in $\mathcal{R}/\mathfrak{p}$, and ϕ the injective inclusion map $\mathcal{R}/\mathfrak{p} \hookrightarrow \mathcal{R}/\mathfrak{p} \subseteq \mathcal{K}(p)$. The salient feature of the mapping $f : p \mapsto f(p)$ in $\mathcal{K}(p)$ is that $f(p) = 0$ if and only if f is contained in $\mathfrak{p}$.

Definition 46. (Regular Function) We then say that this f in $\mathcal{R}$ defines a *regular function* f over $Spec(\mathcal{R})$, which is the mapping $f : Spec(\mathcal{R}) \to \mathcal{R}/\mathfrak{p} \to \mathcal{K}(p)$ given above.

In this way, we can talk about the zeroes of the regular function f, by which we mean those elements p of $Spec(\mathcal{R})$ corresponding to prime ideals $\mathfrak{p}$ which contain f. Beyond this, we can refer to the intersections of the sets of zeroes of two or more regular functions: For $S \subseteq \mathcal{R}$, we can define $V(S) := \{p \text{ in } Spec(\mathcal{R}) \mid f(p) = 0 \text{ for all } f \text{ in } S\}$. Note that $V(\{f\})$ consists of the zeroes of f, and $V(S) = \bigcap_{f \in S} V(\{f\})$.

Definition 47. The *Zariski Topology* designates each $V(S)$, S a subset of $\mathcal{R}$, a closed set, and associates to each such S the open set $Spec(\mathcal{R}) \backslash V(S)$. Because $V(S)$ is itself an intersection of closed sets $V(\{f\})$, each $Spec(\mathcal{R}) \backslash V(S)$ is the union of open sets $Spec(\mathcal{R}) \backslash V(\{f\})$, called the *distinguished open set* U_f of f, which can be thought of as those elements of $Spec(\mathcal{R})$ corresponding to prime ideals in $\mathcal{R}$ which do not contain f. (In less precise but more plainspoken language, these can be thought of as the ideals generated by prime elements which do not divide f, disregarding the zero ideal, which is also prime for any integral domain). These *distinguished open sets* form the basis of the Zariski Topology.

The goal at this point is to associate to this topology a sheaf of commutative rings with unit, and while there are several available (associating the trivial ring to each open set constitutes a valid sheaf, for one), Grothendieck created a sheaf of rings, called the *structure sheaf* of $Spec(\mathcal{R})$, which encodes much of the structure of $\mathcal{R}$ itself.

Definition 48. The *structure sheaf* $\mathcal{O}$: $\mathfrak{Open}(X) \to \mathfrak{Ring}$, for X the topological space formed by topologizing $Spec(\mathcal{R})$ with the Zariski topology and $\mathfrak{Ring}$ the category of commutative rings with unit, is the unique sheaf such that for $U_f = X \backslash V(\{f\})$ the distinguished open set of X associated to the element f of $\mathcal{R}$, $\mathcal{O}(U) := \mathcal{R}[x]/ < f \cdot x - 1 > \cong \mathcal{R}[\frac{1}{f}]$, denoted $\mathcal{R}_f$.

Please note a few things about this association:

1. For $f = 0$, the closed set $V(\{f\})$ is all of $Spec(\mathcal{R})$, in which case $U_f = \varnothing$, so $< f \cdot x - 1 > = < 1 >$, the unit ideal containing the entire ring $\mathcal{R}$. Thus, $\mathcal{O}(U_f) = \{0\}$, the trivial ring.

2. The spectrum of a quotient ring is homeomorphic (under the Zariski Topology) to the spectrum of the original ring without those prime ideals included in the kernel of the quotient mapping. This means that $Spec(\mathcal{O}(U_f)) \simeq U_f$, as the adjunction of $\frac{1}{f}$ creates a ring isomorphic to the adjunction of $\{\frac{1}{f_1}, ..., \frac{1}{f_n}\}$ for $\{f_i\}$ the prime factors of f. Thus, as $\mathcal{R} \subseteq \mathcal{R}$ is generally not considered a prime ideal, the introduction of a multiplicative inverse means that the previously prime ideal $< f_i >$ now also contains $f_i \cdot \frac{1}{f} \cdot \prod_{j \neq i} f_j = 1$, and so $< f_i >$ now generates all of $\mathcal{R}[\frac{1}{f}]$, making it no longer a prime ideal.

3. This suggests a rather natural restriction morphism, which we elaborate on presently: For $f = g \cdot h$ an element of $\mathcal{R}$, the closed set $V(\{f\})$ is clearly $V(\{g\}) \bigcup V(\{h\})$, so contrapositively, $U_f = U_g \bigcap U_h$. How then to define res_{U_g,U_f}? Because both $\mathcal{R}_f$ and $\mathcal{R}_g$ contain canonical copies of $\mathcal{R}$, the image of $\mathcal{R}$ in one maps to the image of $\mathcal{R}$ in the other. But what of $\frac{1}{g}$? Because this must be a homomorphism, $res_{U_g,U_f}(\frac{1}{g}) \cdot g - 1$ must be equal to 0 in $\mathcal{R}_f$ just as $\frac{1}{g} \cdot g - 1 = 0$ in $\mathcal{R}[\frac{1}{f}]$, so $\frac{1}{g} \mapsto \frac{1}{f} \cdot h$ such that $\frac{h \cdot g}{f} - 1 = 0$ as required.

4. This functor, defined over the basic open sets, has not yet given us a complete picture of what the full sheaf must look like. While it is true that a sheaf defined over a base of open sets extends uniquely to (and therefore well-defines) a sheaf over the whole space, this is not immediately obvious, and certainly not to the new student of sheaves. As of yet, we have only laid the groundwork for this extension. We will attempt to fix this now.

Theorem 3.2.1. *The structure sheaf as given is well-defined and unique.*

Proof. Our first step will be to regain our bearings and determine that the sheaf axioms hold in the cases we have already ascribed.

1. Clearly, for the above, if $f = g \cdot h$ and $c = f \cdot d$, then the composition of restriction maps $res_{U_f,U_c} \circ res_{U_g,U_f}$ is equal to res_{U_g,U_c}, as the canonical copy of $\mathcal{R}$ in one will map onto the canonical copy of $\mathcal{R}$ in the other, and $\frac{1}{g} \mapsto h \cdot \frac{1}{f} \mapsto h \cdot (d \cdot \frac{1}{c})$ in either case, as $c = g \cdot d \cdot h$, so $\frac{h \cdot d}{c}$ is algebraically indistinguishable from $\frac{1}{g}$.

2. Furthermore, res_{U_f,U_f} is by inspection the identity map.

3. As for the combined sheaf axiom, we need only show that for each open covering $U_f = \bigcup_{a \in A \subseteq R} U_a$, of U_f, the distinguished open set of an element f of $\mathcal{R}$, that for each family of elements $\{r_a\}_{a \in A}$ with r_a in $\mathcal{R}[\frac{1}{a}]$ such that the restrictions of r_a and r_b agree on restriction to every basic open subset U_c contained within $U_a \bigcap U_b$, then there exists a unique r_f in $\mathcal{R}[\frac{1}{f}] = \mathcal{O}(U_f)$ such that $res_{U_f,U_a}(r_f) = r_a$ for all a in the indexing set A.

   Well, as we've already demonstrated that the restrictions commute, and any intersection of basic open sets is a basic open set itself, we need show only that there exists a unique r_f such that $res_{U_f,U_a}(r_f) = r_a$ for all the a in A, for each family of sections $\{r_a\}_{a\in A}$ such that the restrictions of r_a and r_b agree on $U_a \bigcap U_b$ for all a and b in A. This is sufficient because all basic open subsets contained within these will necessarily be agreed upon by commutative diagram. And, true to form, these restriction morphisms are injective as given, which takes care of the problem of uniqueness. (The map $res_{U_g,U_f} : \mathcal{R}[\frac{1}{g}] \to \mathcal{R}[\frac{1}{f}]$ is isomorphic to the inclusion $\mathcal{R}[\frac{1}{g}] \hookrightarrow \mathcal{R}[\frac{1}{g}][\frac{1}{h}]$, so the preimage of any element under a restriction mapping is necessarily either empty or a single element.)

What then guarantees existence? For this, we must look at the rings and basic open sets themselves. What can we say, *a priori*, about the sets $\{r_a\}$ described above? To begin, $U_f = \bigcup_{a\in A\subseteq R} U_a$ means that we know $U_a \subseteq U_f$ for all a. This means that for every such a, if a prime ideal $\mathfrak{p}$ contains a, it must contain f as well. So, a must divide some power of f, which we can write as $f = a^n \cdot g$ for some g, so we can take res_{U_f,U_a} to be the inclusion $\mathcal{R}[\frac{1}{f}] \hookrightarrow \mathcal{R}[\frac{1}{f}][\frac{1}{g}]$. Thus, even if we don't know a nicely divides f, the morphism can be considered in much the same way regardless.

We now consider the set $\{r_a\}_{a\in A}$, and attempt to constructively prove the existence of an element r_f (which for clarity we will denote without subscript as r) such that r maps to each r_a as required. To begin, we note that r_a factors into $b_a \cdot (\frac{1}{a})^{N_a}$, with b_a in $\mathcal{R}_f$, for some sufficiently large N_a, which means that $a^{N_a} \cdot r_a$ is an element of $\mathcal{R}_f$. Note that we say "in" in this case under the metaphor of ring inclusion, associating $\mathcal{R}_f$ to its image. It would, of course, be more precise to say that $a^{N_a} \cdot r_a$ is contained within the image $res_{U_f,U_a}(\mathcal{R}_f)$. Let us denote for the sake of convenience $res^{-1}_{U_f,U_a}(a^{N_a} \cdot r_a)$ as h_a in $\mathcal{R}_f$.

At this point, we take a slight detour.

Lemma 3.2.2. *Every affine scheme is* quasi-compact*: every open cover of an affine scheme contains a finite subcover. In particular, every open cover of an affine scheme by distinguished open sets contains a finite subcover.*

Proof. (Lemma): We now note that the set $\{a\}_{a\in A} \subseteq \mathcal{R}_f$ must necessarily generate the entire ring $\mathcal{R}_f$ as an ideal, or there would exist some prime ideal $\mathfrak{q}$ in $\mathcal{R}_f$ corresponding to a point q in $\operatorname{Spec}(\mathcal{R}_f)$, here identified with U_f, not covered by the open covering provided. We note also that this correspondence works in both directions: for any set $\{a\}_{a\in A}$ which generate the unit ideal in $\mathcal{R}_f$, the $\{U_a\}$ provide a covering of U_f. Because only finitely many elements are necessary to create 1 in any linear combination, every cover therefore necessarily contains a finite subcover. ∎

We can therefore resort to proving the initial claim for $\{a\}$ finite. All of that is to say that we can take $max(\{N_a\}_{a\in A}) := N$, removing the problematic possibility that, say, $\{N_a\}$ is an infinite increasing sequence. We then recall our definition $h_a := res^{-1}_{U_f,U_a}(a^N \cdot r_a)$ for convenience.

Now, because r_a and r_b agree on all restrictions to distinguished open sets contained in their intersection, and the intersection itself in particular, we have $b^N \cdot h_a = (a \cdot b)^N \cdot r_a = (a \cdot b)^N \cdot r_b = b^N \cdot h_a$.
As we have shown above, $\{a\}$ generates 1 in $\mathcal{R}_f$, and so there must exist some collection $\{e_a\}_{a \in A} \subseteq \mathcal{R}_f$ such that $\sum_{a \in A} e_a \cdot a^N = 1$ in $\mathcal{R}_f$.
Consider now $r := \sum_{a \in A} e_a \cdot h_a$. It is our claim that this is the r we've been looking for.
Clearly, $b^N \cdot r = b^N \cdot \sum_{a \in A} e_a \cdot h_a = \sum_{a \in A} e_a \cdot h_a \cdot b^N$.
But as $h_a \cdot b^N = h_b \cdot a^N$ for every pair $\{a, b\}$ in A,
$b^N \cdot r = \sum_{a \in A} e_a \cdot h_b \cdot a^N = h_b \cdot \sum_{a \in A} e_a \cdot a^N = h_b \cdot 1 = h_b = b^N \cdot r_b$
And so, $(\frac{1}{b})^N \cdot res_{U_f, U_b}(b^N \cdot r) = (\frac{1}{b})^N \cdot res_{U_f, U_b}(b^N \cdot r_b)$,
Which gives us $(\frac{1}{b})^N \cdot b^N \cdot res_{U_f, U_b}(r_b) = (\frac{1}{b})^N \cdot b^N \cdot res_{U_f, U_b}(r) \Rightarrow r = r_b$. Thus, $r = r_b$ for all b by injectivity, and it is therefore the unique element we need to satisfy the sheaf axiom.

■

This is all well and good, as it defines and defends the structure sheaf as such on the distinguished open sets themselves, but how to extend the sheaf to unions of basic sets?

Definition 49. A $\mathfrak{B}$-*sheaf* over a topological space X is a sheaf defined over a basis of open sets $\mathfrak{B}$ of X.

Theorem 3.2.3. *A $\mathfrak{B}$-sheaf over a topological space X extends uniquely to a sheaf over X.*

Proof. In the language of an arbitrary sheaf, we say we extend a $\mathfrak{B}$-sheaf $\mathcal{O}_{\mathfrak{B}}$ defined over a basis of open sets $\mathcal{B}$ of a topological space X to a sheaf $\mathcal{O}$ over the whole topology of X by associating to an arbitrary open set $U \subseteq X$ the ring
$\mathcal{O}(U) := \varprojlim_{V \subseteq U, V \in \mathcal{B}} \mathcal{O}(V)$
$= \{(f_V) \in \prod_{V \subseteq U, V \in \mathcal{B}} \mathcal{O}(V) \text{ such that } res_{V,W}(f_V) = f_W \text{ for all } W \subseteq V \subseteq U;\ V, W \in \mathcal{B}\}$
$= \prod_{V \subseteq U, V \in \mathcal{B}} \mathcal{O}(V)$ modulo agreement on restriction morphisms.
It may be dissatisfying to note that, unwinding definitions, this essentially amounts to defining the extended sheaf in "that unique way that makes it work as a sheaf." Bear in mind, however, that the universal property granted from the inverse limit functor guarantees that the full sheaf $\mathcal{O}$ is well-defined and unique. ■

Definition 50. (Ringed Space) It is worth noting that a scheme is a special case of what is called a *ringed space*, which is to say, topological spaces X equipped with a sheaf $\mathcal{O}$ of commutative rings with unit. Such a space is denoted $(X, \mathcal{O})$.

Note that from now on, we may discuss more than one ringed space at a time, and will denote each sheaf to specify which space it is over. The above $(X, \mathcal{O})$ would become $(X, \mathcal{O}_X)$, with the stalk over x in X denoted $\mathcal{O}_{X,x}$.

3.2.2 Generalizing Beyond the Affine Case: The Grand Scheme of Things

In much the same way as how any n-manifold can be constructed by the gluing together of neighborhoods pulled from $\mathbb{R}^n$, (and additionally, how we use this property to define, evaluate, and overlay manifolds with functions), so too is the relationship between general schemes and their friendlier Affine cousins.

Definition 51. A ringed space $(X, \mathcal{O}_X)$ is called a *scheme* if it is locally affine, which is to say, if, for all points x in X, there exists some open set U_α of X containing x such that the ringed space $(U, \mathcal{O}_U)$ (with $\mathcal{O}_U := \mathcal{O}_X|_U$ the restriction of $\mathcal{O}_X$ to open sets contained within U) isomorphic to the affine scheme $Spec(\mathcal{O}_X(U))$. This is equivalent to saying $X = \bigcup_\alpha U_\alpha$, where U_α is an open set of X and is isomorphic to the affine scheme $Spec(\mathcal{O}_X(U_\alpha)$.

In possession of one or more schemes, it occurs as a natural question how to create more. Perhaps the simplest method is identifying subsets of a scheme $(X, \mathcal{O}_X)$ which are themselves (or are easily made into) locally affine ringed spaces. For instance, as we may notice from the construction of the structure sheaf of an affine scheme, any distinguished open set is itself an affine scheme, with the sheaf restricted in the obvious way. For more complicated schemes, this is not always so simple, although we may bear in mind that every affine open subset itself contains distinguished open sets which are also affine schemes. Note, however, that as affine schemes form a covering of X with distinguished open sets (themselves affine schemes) forming the bases of these sets, that every open subset can be covered with affine subschemes, and that therefore U is what we refer to as an *open subscheme*. Closed subschemes also exist: these are made by a quotient map from an affine open subscheme U_α with kernel an ideal J of $\mathcal{O}_X(U_\alpha)$, thereby associating $V(J)$ as described earlier with $Spec({}^{\mathcal{O}(U_\alpha)}/_J)$, which is precisely that ring having as its spectrum the prime ideals of $\mathcal{O}_X(U_\alpha)$ containing J, obtained by the natural quotient map. This associates $V(J)$ with $Spec(\mathcal{O}_X(U_\alpha)$, providing a mapping which respects the sheaf structure, creating a new scheme in the process.

More simply, we can also disjointly union two schemes together, creating a disconnected scheme containing each of the original schemes as open subschemes. One important aspect of this method is the following:

Lemma 3.2.4. *If the original schemes X and Y are affine, the union $X \amalg Y$ is affine also.*

Proof. Let $(X, \mathcal{O}_X) \cong Spec(A)$ and $(Y, \mathcal{O}_Y) \cong Spec(B)$ be affine schemes, for commutative rings with unit A and B. Now, let ring $C := A \times B$, with addition and multiplication defined coordinate-wise. (The additive identity is $(0_A, 0_B)$, the multiplicative identity is $(1_A, 1_B)$, and so on). The ideals of this ring are the cartesian products of ideals in A and B. Now, for $\mathfrak{p}_A$ any prime ideal of A, $\mathfrak{p}_A \times B$ must be a prime ideal of C. This ideal is proper because $\mathfrak{p}_A$ does not contain all of A. It is furthermore prime because for any element $(a \cdot a', b \cdot b')$ of $\mathfrak{p}_A \times B$, with $a \cdot a'$ contained in $\mathfrak{p}_A$, either a or a' (a, without loss of generality) is in $\mathfrak{p}_A$. And clearly, B contains b. Thus, (a, b) must be contained in $\mathfrak{p}_A \times B$, rendering $\mathfrak{p}_A \times B$ a prime ideal. Therefore, any ideal of the form $\mathfrak{p}_A \times B$ or $A \times \mathfrak{p}_B$, (for $\mathfrak{p}_A$ a prime ideal in A or $\mathfrak{p}_B$ a prime ideal in B), is a prime ideal in C.

We furthermore claim that these are the only prime ideals of C. To prove this, let J be an ideal of A and I an ideal of B. If either is a proper ideal which is not prime, say J, then there exist $a, a' \in A$ such that neither is in J, yet $a \cdot a'$ is. Thus, for any i in I, $J \times I$ must contain $(a, b) \cdot (a', b)$ without either (a, b) or (a', b) being elements of $J \times I$, so this ideal cannot be prime. Thus, we are left only with prime ideals and nonproper ideals. Of course, $A \times B$ is not a proper ideal of C, and so it cannot be a prime ideal either. This leaves us only with the product of a prime ideal and a whole ring or the product of two prime ideals. Suppose then, J and I are prime ideals of their respective rings. Then let $j \in J$ and $i \in I$. As both ideals must be proper, 1_A is not in J, nor is 1_B in I. Clearly, however, $(1_A, i) \cdot (j, 1_B) = (j, i)$ is in $J \times I$, so this ideal cannot be prime either. Thus, as a set, at least, $Spec(C) = Spec(A) \amalg Spec(B)$.

Now we take the affine scheme $(Z, \mathcal{O}_Z) \cong Spec(C)$ and attempt to show that the inclusion map $Spec(A) \amalg Spec(B) \to Spec(C)$; $\mathfrak{p}_A \mapsto (\mathfrak{p}_A \times B)$, $\mathfrak{p}_B \mapsto (A \times \mathfrak{p}_B)$, induces an isomorphism of schemes. For (a, b) an element of C, the distinguished open set $U_{(a,b)}$ is the union of the set of all ideals $\mathfrak{p}_A \times B$ and the set of all ideals $A \times \mathfrak{p}_B$ such that prime ideal $\mathfrak{p}_A$ does not contain a, and prime ideal $\mathfrak{p}_B$ does not contain b.

Consider now $U_{(0,1)}$. $\mathfrak{p}_A$ contains 0 for all $\mathfrak{p}_A$ prime in A, but as prime ideals are necessarily proper, no prime ideal in B contains 1_B. Thus $U_{(0,1)}$ consists of all $A \times \mathfrak{p}_B$, $\mathfrak{p}_B$ prime in B. As this is a distinguished open set of an affine scheme, it must itself be an affine subscheme, isomorphic to the affine scheme $Spec(\mathcal{O}_C(U_{(0,1)}))$, where $\mathcal{O}_C(U_{(0,1)}) = {}^{C[x]}/_{< x \cdot (0,1) - (1,1) >} = \{0\} \times B[1]$, where $\{0\}$ is the trivial ring, the only ring with the additive identity a unit, and $B[1]$ denoting that the image of x is simply 1_B. Perhaps an easier way of viewing this is as the quotient ${}^{A \times B}/_{A \times \{0\}} \cong B$. Thus, B is isomorphic to its image in the mapping above, and, without loss of generality, so is A.

Now, as these images are disjoint (given that prime ideals are necessarily proper, so no two elements of X and Y respectively have the same image), and the union of their image is all of Z, we have given an isomorphism from the disjoint union of affine schemes X and Y onto the scheme Z, showing that the disjoint union also constitutes an affine scheme. ■

Or, to draw off the topological properties of schemes, for $Spec(R) \cong U \cong V$ as schemes, U a subscheme of X and V a subscheme of Y, we can create a new scheme Z via a quotient mapping which *glues* V onto U, joining the topologies at that set. To see how X fits into Z, we take $X \to X \amalg Y$ the obvious inclusion and compose it with the quotient map $X \amalg Y \to {}^{X \amalg Y}/_{\sim}$, where $u \sim v$ if u is in U, and u, v map to the same point in $Spec(R)$ under the pre-established isomorphisms. The union of any basis of X and any basis of Y have images which clearly cover the quotient space and map locally homeomorphically, so the quotient space is still a scheme.

Definition 52. A *locally ringed space* $(X, \mathcal{O})$ is a topological space X affixed with a sheaf of commutative rings with unit $\mathcal{O}$ such that every stalk $\mathcal{O}_x$, for all x in X, is *local* (containing a unique maximal ideal).

Lemma 3.2.5. *All schemes are locally ringed spaces.*

Proof. For $(X, \mathcal{O}_X)$ an affine scheme, x in X, and $\mathfrak{p}_x$ the prime ideal associated to x in $\mathcal{O}_X(X)$, x is contained in the distinguished open set U_f of every f which is not contained in $\mathfrak{p}_x$, and so the restriction of every such f to $\mathcal{O}_{X,x}$ is a unit. This makes the stalk $\mathcal{O}_{X,x}$ the localization of $\mathcal{O}_X(X)$ at $\mathfrak{p}_x$, a local ring. Because this is a (topologically) local property, every point of a scheme contained in an affine subscheme, which must by definition be all of them, must have a local stalk. ■

We can now add a property which contributes greatly to the "niceness" of a scheme, that of being *locally Noetherian*.

Definition 53. A Scheme $(X, \mathcal{O}_X)$ is considered *locally Noetherian* if it admits a covering of affine neighborhoods $X = \amalg_\alpha U_\alpha$ such that $\mathcal{O}_X(U_\alpha)$ is Noetherian for all α. This property also imbues the property that every affine neighborhood V of X has $\mathcal{O}_X(V)$ Noetherian, and that every stalk $\mathcal{O}_{X,x}$ over a point x in X is Noetherian as well, as every quotient of a Noetherian ring is Noetherian, and the adjunction of finitely many formal variables to a Noetherian ring creates a Noetherian ring as well.

Lemma 3.2.6. *If R is Noetherian, every subset of $Spec(R)$ is quasi-compact.*

Proof. If we can show that every covering by basic open subsets has a finite subcover, quasi-compactness will hold. Take then a subset $\{p_\alpha\}$ of $Spec(R)$ corresponding to prime ideals $\mathfrak{p}_\alpha$ of R. We want to show that for every set of elements $\{f_\beta\}$ in R such that for every α, there is some f_{β_α} in R such that $\mathfrak{p}_\alpha$ does not contain f_{β_α}, we can remove all but finitely many $\{f_\beta\}$ without removing that property.

Consider the ideal generated by $\{f_{\beta_\alpha}\}$, which must not be contained in $\mathfrak{p}_\alpha$ for any α. As R is Noetherian, there is some finite set of finite linear combinations of $\{f_{\beta_\alpha}\}$ which generate this ideal, and so we can take to be $\{f_\gamma\}$ to be the necessarily finite subset of $\{f_{\beta_\alpha}\}$ which makes a non-zero contribution to one of the above linear combinations. Then the ideal generated by $\{f_\gamma\}$ is still not contained by any $\mathfrak{p}_\alpha$ for any α, and so there is some f_γ not contained in $\mathfrak{p}_\alpha$ for each α, and so the set $\{U_{f_\gamma}\}$ provides a finite subcover of $\{f_\beta\}$. ■

3.3 Morphisms of Schemes

The attentive reader may note that the above constructions rely on mappings which, as of yet, have not been rigorously defined. Let us take a moment to fix that.

Definition 54. The following construction provides a *morphism of schemes.*

Given that a scheme $(X, \mathcal{O}_X)$ consists of two structures, the topological space X and the overlaid sheaf of rings $\mathcal{O}_X$, it stands to reason that a mapping of schemes could be determined by where it sends the underlying points and what it does to the structure sheaf. For this reason, we break down the map $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ into a pair of mappings, $(\psi, \psi^{\#})$; $\psi : X \to Y$ a continuous mapping, and $\psi^{\#} : \mathcal{O}_Y \to \psi^*\mathcal{O}_X$ a natural transformation of sheaves *over* Y (morphism of contravariant functors).

The defining characteristic of a continuous mapping of topological spaces $\psi : X \to Y$ is that it induces a mapping of open sets in Y to open sets of X, $Y \supseteq W \mapsto \psi^{-1}(W) \subseteq X$. We can easily categorize the set of open sets over X by making the sets themselves objects and the morphisms between them inclusions, resulting in the category $\mathbb{O}(X)$, with $\mathbb{O}(Y)$ defined analogously. In this perspective, the mapping ψ induces a covariant functor $\psi^{-1} : \mathbb{O}(Y) \to \mathbb{O}(X)$ which respects inclusion.

At the risk of overcomplicating a relatively simple construction, we can now construct a sheaf of rings over Y by composition, defining $\psi^*\mathcal{O}(X) : \mathbb{O}(Y) \to (RING)$ as $\psi^*\mathcal{O}_X := \mathcal{O}_X \circ \psi^{-1}$. The reason for making this mapping into a functor is that, under this perspective, we can consider $\phi^{\#}$ a natural transformation of contravariant functors (which, we may recall, is precisely what sheaves are). This natural transformation can be thought of as a collection of ring homomorphisms $\{\psi^{\#}_W : \mathcal{O}_Y(W) \to \psi^*\mathcal{O}_X(W)\}_{W \subseteq Y open}$ which commute with the restriction morphisms imposed by the sheaves. Please note that, as sheaves are contravariant, although the map is from X to Y, the induced ring homomorphisms are from the rings over Y to the rings over X.

It may be more comfortable to consider this from the opposite perspective: given a ring homomorphism $R \to A$, we can recover a map $Spec(A) \to Spec(R)$ associating to every prime ideal in A its preimage in R. (Recall that we do not by convention consider the trivial mapping to be a homomorphism unless A is the trivial ring, requiring that $1_R \mapsto 1_A$, eliminating the possibility that the preimage of a prime ideal in A might contain the entirety of R). Thus, it might be just as valid to consider a mapping of schemes $X \to Y$ as a collection of ring homomorphisms linking $\mathcal{O}_Y \to \mathcal{O}_X$, inducing a reverse mapping of prime ideals, which we then consider the points of the schemes, as it would be to take the reverse perspective.

We impose one further restriction on such a mapping ψ: Let $p \overset{\psi}{\mapsto} q$, for p in X and q in an open set W of Y. Then for f a section of $\mathcal{O}_Y(W)$, f vanishes at q if and only if $\psi^*(f)$ in $\psi^*\mathcal{O}_X(W) = \mathcal{O}(f^{-1}(W))$ vanishes at p.

We now take a moment to further explore the relationship between mappings of rings and mappings of schemes, using educational exercises 2.4, 2.16, and 2.17 laid out in [Hartshorne].

Theorem 3.3.1. ([Hartshorne] Exercise 2.4)*: For X, $Spec(A)$ schemes with $Spec(A)$ affine, the mapping $\alpha : Hom_{\mathfrak{Sch}}(X, Spec(A)) \to Hom_{\mathfrak{Ring}}(A, \mathcal{O}_X(X))$ associating to every morphism of schemes $f : X \to Spec(A)$ the induced homomorphism of rings $\varphi_f : A \to \mathcal{O}_X(X)$ is bijective.*

Proof. Take $\{Spec(B_\beta)\}$, the set of all affine subsets of X (not only a cover, the whole basis of the topology of X!). Specifying a map $f : X \to Spec(A)$ is equivalent to specifying a set of maps $\{f_\beta : Spec(B_\beta) \to Spec(A)\}_\beta$, modulo that these mappings must agree on all glued intersections, well-defining the mapping into X. This is equivalent to a set of maps $\{\varphi_{f,\beta} : A \to B_\beta\}_\beta$ such that the preimages of two prime ideals $\mathfrak{p}_\beta \subseteq B_\beta, \mathfrak{p}_{\beta'} \subseteq B_{\beta'}$ agree whenever $\mathfrak{p}_\beta$ and $\mathfrak{p}_{\beta'}$ correspond to the same point in X. But this set $\{\varphi_{f,\beta}\}$ is simply a mapping from A into the projective limit $\varprojlim_{\beta} B_\beta$, which was our original definition for $\mathcal{O}_X(X)$. ■

Theorem 3.3.2. [Hartshorne] 2.16*: Given a scheme $(X, \mathcal{O}_X)$ with a global section f, the set X_f of points x in X such that the restriction of f to the stalk $\mathcal{O}_{X,x}$ of x is not contained within the maximal ideal $\mathfrak{m}_x$ is an open subscheme of X, and if X is quasicompact and admits some affine cover $\{U_\alpha\}$ such that the pairwise intersection $U_\alpha \bigcap U_{\alpha'}$ is quasicompact, then $\mathcal{O}_X(X_f) \cong \mathcal{O}_X(X)[\frac{1}{f}]$.*

Proof. We begin by looking at U, an open affine subscheme of X, with $\mathcal{O}_X(U) = B$. We set $res_{X,U}(f) = \bar{f}$, and as any restriction to $\mathcal{O}_x$ for x in U will have to factor through $\bar{f}$, $X_f \bigcap U = U^{\bar{f}}$ (expressing the same notion as X_f, not the distinguished open set of f). $U^{\bar{f}}$ contains exactly those elements x of U such that there exists a distinguished open set U_g of U containing x with $res_{U,U_g}(\bar{f})$ a unit in $\mathcal{O}_X(U_g)$. However, every distinguished open set on which the restriction of $\bar{f}$ is a unit is necessarily contained within the distinguished open set $U_{\bar{f}}$ of f, and so $U^{\bar{f}}$ is necessarily contained within $U_{\bar{f}}$. But every restriction of $\bar{f}$ to the stalk of $\mathfrak{m}_x$ for x in $U_{\bar{f}}$ is also a restriction of $res_{U,U_{\bar{f}}}(\bar{f})$, which is a unit. Thus, the two sets are identical. $U^{\bar{f}} = U_{\bar{f}}$.

Thus, $X_f = \bigcup_\alpha U_{\alpha, res_{X,U_\alpha}(f)}$, the union of the distinguished open sets of $res_{X,U_\alpha}(f)$ in each U_α, and is an open subscheme of X. □

We now examine the case where X is quasi-compact, and claim that if a global section a satisfies $res_{X,X_f}(a) = 0$, then there exists some $n > 0$ such that $f^n \cdot a = 0$ in $\mathcal{O}_X(X)$.

Given the limit definition of a sheaf over arbitrary open sets, $res_{X,X_f}(a) = 0$ if and only if $res_{X,U_{\alpha,res_{X,U_\alpha}(f)}}(a) = 0$ for every U_α in some affine cover, which we can take to be finite. (For clarity, we denote $res_{X,U_\alpha}(f)$ as f_α and $res_{X,U_\alpha}(a)$ as a_α from now on.) This is, in turn, only true if the injection $res_{U_\alpha,U_{\alpha,f_\alpha}}(a_\alpha) = 0$ for every α. This means that a_α is in the ideal $< Z \cdot f_\alpha - 1 >$ in $\mathcal{O}_X(U_\alpha)[Z]$, which occurs when $f_\alpha^{n_\alpha} \cdot a_\alpha = 0$. We then take $\max_\alpha(n_\alpha)$ to be n. Then $res_{X,U_\alpha}(f^n \cdot a) = 0$ for all α, which makes it exactly 0 by the sheaf axioms.□

We now claim that for b a section over X_f, there exists some $N > 0$ such that $f^N \cdot b$ is in the image of res_{X,X_f}.

We again examine the restrictions $res_{X_f,U_\alpha \bigcap X_f}$, which we now know to be $res_{X_f,U_{\alpha,f_\alpha}}$. $res_{X_f,U_{\alpha,f_\alpha}}(b) = \frac{b_\alpha}{f_\alpha^{n_\alpha}}$, with some slight abuse of notation, for some b_α in $\mathcal{O}_X(U_\alpha)$, some whole number n_α, and f_α as above. We have specified $\{U_\alpha\}$ as a finite subcover of the affine cover such that $U_\alpha \bigcap U_{\alpha'}$ is quasicompact for any two sets in the cover. As there are finitely many α in our finite subcover, we can replace b_α with $b_\alpha \cdot f^{n-n_\alpha}$, for $n = \max_\alpha(n_\alpha)$, and in so doing, get $res_{X_f,U_{\alpha,f_\alpha}}(b) = \frac{b_\alpha}{f_\alpha^n}$.

We next consider the restrictions of $b_\alpha, b_{\alpha'}$ to $U_\alpha \bigcap U_{\alpha'}$, which we denote $b'_\alpha, b'_{\alpha'}$. Their restrictions to $X_f \bigcap (U_\alpha \cap U_{\alpha'})$ must agree, with $U_\alpha \bigcap U_{\alpha'}$ quasi-compact, and so we can use the result from the previous subsection of this proof to say that as $(b'_\alpha - b'_{\alpha'})$ must vanish on the intersection, there exists some n' such that $f^{n'} \cdot (b'_\alpha - b'_{\alpha'}) = 0$ in $\mathcal{O}_X(U_\alpha \bigcap U_{\alpha'})$. We now have two sets of elements $\{b_\alpha\}$ and $\{b'_\alpha\}$. $\{b_\alpha\}$ are sections associated to each set of an affine cover whose restrictions agree on pairwise intersections, and so there exists a unique global section c which restricts to each b_α. Likewise, $\{b'_\alpha\}$ are associated to each open set of an affine cover of X_f and agree on pairwise intersections, and so by construction, we can take $f^{n+n'} \cdot b$ as the unique element of $\mathcal{O}_X X_f$ restricting to each b'_α. However, because b_α restricts to b'_α, c must restrict to $f^{n+n'} \cdot b$ on X_f. We then take $N = n + n'$, which gives us $f^N \cdot b$ in the image of res_{X,X_f}.□

Now, as the restriction of f to X_f has a multaplicative inverse, we can uniquely extend the restriction map res_{X,X_f} to a morphism $\mathcal{O}_X(X)[\frac{1}{f}] \to \mathcal{O}_X(X_f)$. Any element of $\mathcal{O}_X(X)[\frac{1}{f}]$ can be written as $\frac{c}{f^n}$. Take an element of the kernel of this mapping. By the above, there exists some m such that $f^m \cdot c = 0$ in $\mathcal{O}_X(X)$, which necessitates that $\frac{c}{f^n}$ be zero in $\mathcal{O}_X(X)[\frac{1}{f}]$. This gives injectivity.

We also have just shown that for any element b of $\mathcal{O}(X_f)$, there is some N such that $f^N \cdot b$ is the restriction of some c in $\mathcal{O}_X(X)$. However, this means that $\frac{c}{f^N}$ in $\mathcal{O}_X(X)[\frac{1}{f}]$ must map to b, which yields surjectivity.

Thus, we are given an isomorphism of rings $\mathcal{O}_X(X_f) \cong \mathcal{O}_X(X)[\frac{1}{f}]$. ■

Lemma 3.3.3. ([Hartshorne] Exercise 17a)*: Let $f : X \to Y$ be a morphism of schemes. Then if there exists an open cover $\{U_\alpha\}$ of Y such that the induced homomorphism of rings $\varphi_\alpha : \mathcal{O}_Y(U_\alpha) \to \mathcal{O}_X(f^{-1}(U_\alpha))$ is an isomorphism for all α, then f is an isomorphism of schemes.*

Proof. We begin by taking an open affine cover $\{V_\beta\}$, and an open affine cover $\{W_{\alpha,\beta}\}$ of $U_\alpha \bigcap V_\beta$ of distinguished open sets of V_β. As the map $f^{-1}(U_\alpha) \to U_\alpha$ is an isomorphism, we can identify via isomorphism $f^{-1}(W_{\alpha,\beta}) \to W_{\alpha,\beta}$ as well. We note that the sets $\{f^{-1}(W_{\alpha,\beta})\}, \{W_{\alpha,\beta}\}$ are each an open affine cover of X and Y respectively, identified bijectively and isomorphically. As a scheme is defined by its construction by gluing open affine sets together, and the correspondance of gluings is provided by the bijective association between the covers, we get $X \cong Y$. ■

Theorem 3.3.4. ([Hartshorne] Exercise 2.17b)*: A scheme $(X, \mathcal{O}_X)$ is affine if and only if there exist a finite set of global sections $\{f_1, \dots, f_n\}$ such that the open subsets X_{f_i} are affine, and $f_1, \dots, f_n$ generate the unit ideal in $\mathcal{O}_X(X)$.*

Proof. (Sufficiency): From Theorem 3.3.1, we know that the isomorphism of rings $\mathcal{O}_X(X) \to \mathcal{O}_X(X)$ uniquely corresponds to a morphism of schemes $X \to Spec(\mathcal{O}_X(X))$. We claim this is an isomorphism of schemes. We know $f_1, \dots, f_n$ generate $\mathcal{O}_X(X)$, and so the distinguished open sets $\{U_{f_i}\}$ form a finite affine cover of $Spec(\mathcal{O}_X(X))$ The preimage of U_{f_i} is simply X_{f_i}, which we have given as affine. We also know from Theorem 3.3.2 that $U_{f_i} \cong X_{f_i} \cong \mathcal{O}_X(X)[\frac{1}{f_i}]$. As an affine scheme is determined by its global ring, we know that U_{f_i} and X_{f_i} are *isomorphic* as affine schemes, but we do not know if the given mapping is an isomorphism. Fortunately, we know from Theorem 3.3.1 that there is exactly one morphism of an affine scheme to itself which induces a given isomorphism on its global ring, and that is an isomorphism itself. Thus, by Lemma 3.3.3, we are done.

In the other direction, for X affine, the section 1 in $\mathcal{O}_X(X)$ clearly generates the unit ideal, and so the condition is necessary as well as sufficient for affinity. ■

Lemma 3.3.5. *For affine schemes $X \cong Spec(A)$, $Y \cong Spec(B)$, and $Z \cong Spec(C)$, with morphisms $f : Y \to X$ and $g : Z \to X$, the fiber product $Y \underset{X}{\times} Z$ is well-defined as an affine scheme and isomorphic to $Spec(B \underset{A}{\otimes} C)$.*

Proof. The construction and bilinearity of the tensor product $B \underset{A}{\otimes} C$ make its prime ideals exactly those such that their projection into B and C coincide under the maps f and g, which gives isomorphism. As the tensor product is, in this case, a ring itself, we are given affinity. ■

Having associated to a map of schemes a set of ring homomorphisms in the opposite direction, we can now examine an interesting feature of the points of a scheme. Take a scheme $(X, \mathcal{O}_X)$ containing a point x. We have already discussed how X must be locally-ringed, and as such, we can talk about $\mathfrak{m}_{X,x} \subseteq \mathcal{O}_{X,x}$, the unique maximal ideal of the stalk over x. Recall, from our definition of elements of a ring as quasi-functions over its spectrum, the concept of a *residue field*, the field formed by a quotient map with a maximal ideal as its kernel. Given that $\mathcal{O}_{X,x}$ is by definition local, we can associate to it the unique residue field $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$, which we denote $\mathfrak{K}(x)$.

Now, a mapping of schemes $\psi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $x \mapsto y$, induces the map of sheaves $\psi^\# : \mathcal{O}_Y(U) \to \mathcal{O}_X(\psi^{-1}(U))$ for U any open set of Y. This means that $\psi^\#$ associates to any such y a collection of morphisms of rings $\{\psi_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\psi^{-1}(U))\}_{U \ni y}$, and that each $\psi^{-1}(U)$ necessarily contains x as well. The limit property of stalks over the points x and y allows us to determine from these morphisms $\{\psi_U\}_{y \in U}$ a map of stalks (morphism of rings) $\psi_y : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. It is worth noting that these mappings of stalks capture many local properties of scheme morphisms, and, taken together, uniquely determine the morphism itself.

To examine a particular point x in $(X, \mathcal{O}(X))$, however, we may wish to look at a mapping directly to this point and nowhere else. From the perspective of X as a set, this may seem uninteresting, but the associated scheme structure makes it worth our while. The aforementioned map $(X, \mathcal{O}(X)) \to \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ provides a ready-made morphism of rings, inducing the morphism of schemes $Spec(\mathfrak{K}(x)) \to X$, $< \mathfrak{o} > \mapsto x$. The advantages of examining this mapping stem from that, by definition, the field $\mathfrak{K}(x)$ has a unique prime (and therefore maximal) ideal, which easily maps onto x without requiring further specification.

The reader may notice, however, that the niceness of this map is not unique to $Spec(\mathfrak{K}(x))$. In fact, any field $\mathfrak{K}$ which can be mapped to from $\mathcal{O}_{X,x}$ with $\mathfrak{m}_x$ as the kernel satisfies this property. What we have described here, however, is a map $\mathcal{O}_{X,x} \to \mathfrak{K}$ which can be factored through $\mathcal{O}_{X,x} \to \mathfrak{K}(x) \to \mathfrak{K}$, and as the kernel of a ring homomorphism must be an ideal, such a factoring would necessarily have either

$< 0 >$ or $\mathfrak{K}(x)$ as the kernel of its final step. The latter would require that the mapping be the trivial mapping, which we consider a homomorphism only onto the trivial ring, which is not a field and cannot be $\mathfrak{K}$. Therefore, we can conclude that $\mathfrak{K}(x)$ maps isomorphically onto its image in $\mathfrak{K}$, which with slight abuse of notation, we can associate to an inclusion (extension) of fields $\mathfrak{K}(x) \subseteq \mathfrak{K}$.

Definition 55. This mapping, $\mathcal{O}_{X,x} \to \mathfrak{K}$, or rather, the map $Spec(\mathfrak{K}) \to \{x\} \subseteq X$ of schemes which induces it, constitutes what we call a $\mathfrak{K}$-*rational point* in X.

Definition 56. If $\mathfrak{K}$ is separably closed, we call a $\mathfrak{K}$-rational point a *geometric point*.

Now, much as the residue field $\mathfrak{K}(x)$ has the distinction of being the uniquely smallest field such that x is $\mathfrak{K}(x)$-rational, there exists a uniquely smallest field $\mathfrak{L}$ such that $Spec(\mathfrak{L}(x)) \to \{x\}$ is a geometric point in X. As x is $\mathfrak{K}$-rational for fields $\mathfrak{K}$ containing $\mathfrak{K}(x)$, this is, of course, simply the smallest separably closed field $\mathfrak{L}$ containing $\mathfrak{K}(x)$, which is just $\overline{\mathfrak{K}(x)}$, the separable closure of $\mathfrak{K}(x)$, a concept which we applied to the Galois Theory problem above.

We can now give formal definitions of important properties which morphisms of schemes might exhibit, including the earlier-referenced *étale map*.

Definition 57. $f : X \to Y$ a map of schemes is *affine* if, for all y in Y, there exists some affine neighborhood U containing y such that $f^{-1}(U) \subseteq X$ is affine.

Definition 58. $f : X \to Y$ a map of schemes is *finite* if, for all y in Y, there exists some affine neighborhood U containing y such that $f^{-1}(U) \subseteq X$ affine, and the map of rings $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ gives $\mathcal{O}_X(f^{-1}(U))$ the structure of a finite $\mathcal{O}_Y(U)$-module. (This is to say, if there exist finitely many elements $\{r_1, \ldots, r_n\}$ such that $r_1 \cdot \mathcal{O}_Y(U) + \cdots + r_n \cdot \mathcal{O}_Y(U)$ spans $\mathcal{O}_X(f^{-1}(U))$).

Definition 59. $f : X \to Y$ a map of locally Noetherian schemes is *étale* if for all y in Y and all x in $f^{-1}(\{y\})$, there exists some affine neighborhood U containing y and affine V containing x such that V is contained within $f^{-1}(U)$, and the map of rings $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ has the form $\mathcal{O}_Y(U) \to {}^{\mathcal{O}_Y((U))[x]}\!/_{< h >_{\mathfrak{B}}}$, for ${}^{\mathcal{O}_Y((U))[x]}\!/_{< h >_{\mathfrak{B}}}$ the localization of ${}^{\mathcal{O}_Y((U))[x]}\!/_{< h >}$ at some prime ideal $\mathfrak{B}$ and h a monic polynomial such that h' is invertible in ${}^{\mathcal{O}_Y((U))[x]}\!/_{< h >}$.

Definition 60. $f : X \to Y$ a map of locally Noetherian schemes is *finite étale* if f is both a finite map and an étale map. A scheme X equipped with a finite étale map onto scheme Y is called an *étale covering* of Y. Such a covering is denoted (X, f), $X \xrightarrow{f} Y$, or simply ${}^{X}\!/_{Y}$.

From these definitions, it is true by inspection that all finite morphisms (and therefore all finite étale morphisms) are affine.

Please note that the property of being locally Noetherian is so important in simplifying our discussion of étale maps that, from this point forward, schemes may be assumed to be locally Noetherian. For formal statements, we may include this provision explicitly, but the assumption carries even when not stated.

There are, for each of these properties, equivalent definitions which are much more useful, but these definitions are standard. However, as these equivalences are nontrivial, we take it upon ourselves to show them here. First, however, we must demonstrate the following particularly useful property of affine morphisms:

Lemma 3.3.6. *Given an affine morphism of locally Noetherian schemes $f : X \to Y$ and an open affine subset $U \subseteq Y$, the restriction $f|_{f^{-1}(U)}$ of f to $f^{-1}(U)$ is also affine.*

Proof. f affine means that every point is included in some open affine $Y_\alpha \subseteq Y$ such that $f^{-1}(Y_\alpha)$ is open affine in X. Take these $\{Y_\alpha\}_{\alpha \in A}$ as an open affine cover of Y. As the Y_α's cover Y, the set $\{Y_\alpha \bigcap U\}_{\alpha \in A}$ must be an open (not necessarily affine!) cover of U. We now fix $u \in U$. Then there exists $\alpha \in A$ such that $Y_\alpha \bigcap U$ contains u. Suppose this $Y_\alpha \cong Spec(R_\alpha)$ as an affine scheme. Then, because affine subsets form a basis of Y, and designated open subsets (themselves affine open subsets) form the basis of Y_α, there must exist some designated open subset $U_{a_{\alpha,u}}$, $a_{\alpha,u} \in R_\alpha$, such that u is contained within $U_{a_{\alpha,u}}$ and $U_{a_{\alpha,u}}$ is contained within the intersection $Y_\alpha \bigcap U$. Now, f is, of course, topologically continuous, and we've already established that $f^{-1}(Y_\alpha)$ is affine in X, so the map $f|_{f^{-1}(Y_\alpha)} : f^{-1}(Y_\alpha) \to Y_\alpha$ is simply a morphism of affine schemes.

We then examine $f^{-1}(U_{a_{\alpha,u}}) \subseteq f^{-1}(Y_\alpha)$. Consider $\varphi_\alpha : R_\alpha \to \mathcal{O}_X(f^{-1}(Y_\alpha))$, the Ring homomorphism induced by the map $f|_{f^{-1}(Y_\alpha)}$. Specifically, note that a prime ideal $\mathfrak{p}$ in $\mathcal{O}_X(f^{-1}(Y_\alpha))$ contains φu if and only if the preimage $\varphi_\alpha^{-1}(\mathfrak{p})$ of that ideal contains u. Thus, the designated open set $V_{\alpha,\varphi(u)}$ of $f^{-1}(Y_\alpha))$ is exactly the preimage of $U_{a_{\alpha,u}}$ under f, so $f|_{f^{-1}(U)}$ is locally affine at u. And, since this is true without loss of generality for all such u, we can say $f|_{f^{-1}(U)}$ is affine. ∎

This is of particular importance in the following Theorem, also regarding affine morphisms:

Theorem 3.3.7. *A morphism of schemes $f : X \to Y$ is affine if and only if for every open affine U in Y, its preimage $f^{-1}(U)$ is open affine in X.*

Proof. (Necessity): Let us begin with the case where Y is an affine scheme, and generalize from there.

Let $Y \cong Spec(R)$ be an affine scheme. f is affine, so there must exist an affine cover $\{U_\alpha\}$ of Y, with $U_\alpha \cong Spec(R_\alpha)$, such that $f^{-1}(U_\alpha)$, which we denote V_α, is affine for all α. We now fix a point u_α in U_α. Because distinguished open sets form the basis of affine schemes, there is some section r_α in R such that the distinguished open set U_r contains u_α and is contained within U_α. However, because U_α contains U_r, we can associate U_r with the distinguished open set $U_{r'_\alpha}$ of r'_α, the restriction of r_α to Y_α. Because we know $V_\alpha \to U_\alpha$ a morphism of affine schemes, we know the preimage of $U_{r'_\alpha}$ is a distinguished open set of V_α, which is also affine.

We now consider an arbitrary global section q in R under the induced morphism of rings $\hat{f} : R \to \mathcal{O}_X(X)$. If f restricts to a unit of the stalk $\mathcal{O}_{Y,y}$, then for every x in the preimage of y under f, $\hat{f}(q)$ must restrict to a unit of the stalk $\mathcal{O}_{X,x}$. From this, we can see that $f^{-1}(Y_r)$ contains $X_{\hat{f}(r)}$. From this, we see that $X_{hatf(r_\alpha)}$ is exactly $f^{-1}(Y_{r_\alpha})$, or the preimage of $U_{r'_\alpha}$, which we have shown to be affine.

Because Y is affine and therefore quasicompact, we can take the affine cover $U_{r'_\alpha}$ of Y to be finite. This means that as no point in Y is not contained in the finite union of these sets, no prime ideal of R is not contained within the ideal generated by r_α, with r_α restricting to $r_{\alpha'}$, and so a linear combination of these $\{r_\alpha\}$ is equal to 1 in R. The image of this linear combination must also be 1 in $\mathcal{O}_X(X)$, and so the finite set $\{\hat{f}(r_\alpha)\}$ generate the unit ideal in $\mathcal{O}_X(X)$, with $X_{\hat{f}(r_\alpha)}$ affine for all α in the finite cover. We therefore conclude by Theorem 3.3.4 that X is affine.

Expanding now to the general case, for $f : X \to Y$ an affine map of schemes, we simply take any open affine set U in Y, and we are given by Lemma 3.3.6 that the map $f : f^{-1}(U) \to U$ is an affine morphism onto an affine set. From the above, we then conclude $f^{-1}(U)$ to be affine. ■

Theorem 3.3.8. *A morphism of locally Noetherian schemes $f : X \to Y$ is finite if and only if for every affine open subscheme U of Y, the preimage $f^{-1}(U)$ is an affine subscheme of X and the induced mapping $\hat{f} : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ gives $\mathcal{O}_X(f^{-1}(U))$ the structure of a finitely-generated $\mathcal{O}_Y(U)-$module.*

Proof. If f exhibits this property, then any affine cover of Y will satisfy the conditions necessary to define f as a finite morphism. To show necessity, however, we first assume that f is finite and then take an affine cover $\{U_\alpha\}$ of Y such that $f^{-1}(U)_\alpha$, which we denote as V_α, is affine and $\mathcal{O}_X(V_\alpha)$ is given the structure of a finitely-generated $\mathcal{O}_Y(U_\alpha)-$module. From here, we denote for convenience $U_\alpha \cong Spec(A_\alpha)$, $V_\alpha \cong Spec(B_\alpha)$. We now consider an affine open subscheme of U_α. U_α and V_α are quasicompact, as the spaces X and Y are locally Noetherian. We already know that for a_α in A_α and $\hat{f}_\alpha$ the induced homomorphism of rings $A_\alpha \to B_\alpha$, the distinguished open set $V_{\hat{f}_\alpha(a_\alpha)} = f^{-1}(U_{a_\alpha})$. Then an element b in B_α can be written as $b_\alpha = \sum_{i=1}^{n} \hat{f}_\alpha(a_{\alpha,i}) \cdot b_{\alpha,i}$. As any element of $V_{\hat{f}_\alpha(a_\alpha)}$ can be written as $\frac{b}{\hat{f}_\alpha(a_\alpha)^N}$ for some N, we can write that element as $\frac{b}{\hat{f}_\alpha(a_\alpha)^N} = \sum_{i=1}^{n} \hat{f}_\alpha(\frac{a_{\alpha,i}}{a_\alpha^N}) \cdot b_{\alpha,i}$.

We have now shown that every affine subset contains a small enough distinguished open set surrounding any given point which satisfies this property. We now replace for notational convenience the cumbersome double subscripts and reduce to the case where $f : X \to Y$ is a finite map of affine schemes and seek to show that for $X \cong Spec(B)$, $Y \cong Spec(A)$, $\hat{f}$ gives B the structure of a finitely-generated $A-$morphism.

We already know from f being finite that there exist some distinguished open sets U_a, $V_{\hat{f}(a)} = f^{-1}(U_a)$ and some finite list of m elements $\{b_i\}$ in B such that any element of $B[\frac{1}{\hat{f}(a)}]$, which we call $\frac{b}{\hat{f}(a)^N}$ (with b in B) can be written as $\frac{b}{\hat{f}(a)^N} = \sum_{i=1}^{m} \frac{b_i}{\hat{f}(a)^{n_i}} \cdot \hat{f}(a_i)$, for $\{n_i\}$ fixed. But then such a b can simply be written $b = \sum_{i=1}^{m} b_i \cdot \hat{f}(a_i \cdot a^{N-n_i})$. To avert the problem that might arise if some n_i were greater than N, we note that for arbitrarily large N, there exists some $\{a_i\}$ which allow us to write $\frac{b}{\hat{f}(a)^N}$ with this linear combination, and thus, the problem disappears. ■

Theorem 3.3.9. *For $f : X \to Y$ a morphism of locally Noetherian affine schemes such that $X \cong Spec(A)$ and $Y \cong Spec(B)$ and f has the property that the induced map of rings $\widehat{f} : B \to A$ takes the form $B \to {}^{B[x]}/_{< h >}$, for h a monic polynomial such that h' is invertible in ${}^{B[x]}/_{< h >}$, then the restriction of f to any distinguished open subset $U_a \to f(U_a)$ has this property as well.*

Proof. We begin by noting that $f(U_a) = \{\widehat{f}^{-1}(\mathfrak{p}_\alpha)\}$, where p_α varies over all prime ideals in A not containing a (excusing the abuse of notation which identifies elements of $Spec(R)$ with their corresponding prime ideals in R). We let b be in B such that $\widehat{f}(b) = a$. Then a is in p_α if and only if b is contained in the prime ideal $\widehat{f}^{-1}(\mathfrak{p}_\alpha)$, and so $f(U_a) = V_b$, the distinguished open subset of b in Y.

It remains, then, to show that in the induced map $B[\frac{1}{b}] \to {}^{B[x]}/_{< h >}[\frac{1}{a}]$, ${}^{B[x]}/_{< h >}[\frac{1}{a}]$ can be written as ${}^{B[\frac{1}{b}][x]}/_{< g >}$, for g a monic polynomial with g' invertible in the target. As $b \mapsto a$, we can write ${}^{B[x]}/_{< h >}[\frac{1}{a}]$ as ${}^{B[\frac{1}{a},x]}/_{< h >} = {}^{B[\frac{1}{b},x]}/_{< h >}$, and the adjunction of $\frac{1}{a}$ does nothing to change the invertibility of h', and so we can take $g = h$ and we are done. ■

The following corollaries follow sufficiently directly from Theorem 3.3.9 that we omit their proofs:

Corollary 3.3.10. *For $f : X \to Y$ an affine morphism of locally Noetherian affine schemes such that $X \cong Spec(A)$ and $Y \cong Spec(B)$ and f has the property that the induced map of rings $\widehat{f} : B \to A$ takes the form $B \to {}^{B[x]}/_{< h >}$, for h a monic polynomial such that h' is invertible in ${}^{B[x]}/_{< h >}$, then the restriction of f to $f^{-1}(V_b) \to V_b$ for V_b a distinguished open subset of B has this property as well.*

Corollary 3.3.11. *Given an étale morphism of locally Noetherian schemes $f : X \to Y$, the restriction of f to $U \to f(U)$ is also an étale morphism of locally Noetherierean schemes, for U any open subscheme of X.*

Chapter 4

The Étale Fundamental Group

4.1 Étale Coverings

4.1.1 Étale Coverings as a Category

Definition 61. Suppose we fix a connected, locally Noetherian scheme $(S, \mathcal{O}_S)$ (connected in the sense that it cannot be decomposed into the disjoint union of two nonempty open sets). Then there exists a *Category of Étale Coverings of* $(S, \mathcal{O}_S)$, denoted ${}^{Et}/_S$, whose objects are schemes equipped with finite étale maps onto S and whose morphisms are morphisms of schemes which preserve the equipped étale mappings onto S.

Definition 62. This is to say, a *morphism of objects* $X \xrightarrow{f_X} S$ and $Y \xrightarrow{f_Y} S$ of ${}^{Et}/_S$ is a morphism of schemes $g : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f_X"'] & Y \arrow[d, "f_Y"] \\ & S \end{tikzcd}$$

Definition 63. An *automorphism* of an object $X \xrightarrow{f} S$ in $Ob({}^{Et}/_S)$ is a morphism of objects $X \to X$ which is invertible. The group of all automorphisms of the object $X \xrightarrow{f} S$ is denoted $Aut({}^{X}/_S)$ or $Aut(X)$.

Before proceeding further, there are a few results which will be very helpful to us as we move onward, but whose proofs are made much less onerous (and shorter!) by the use of alternative definitions for many of the properties of morphisms we have examined. Rather than attempting to show equivalence of definitions or working around our limitations, we will simply state the results with reference to more thorough resources for the curious reader:

Lemma 4.1.1. *[Stacks], Lemmata 34.3 and 34.4: Finite étale morphisms are stable under pullback.*

Lemma 4.1.2. *[SGAI], Proposition 3.1: For $Y \xrightarrow{f} S$ a finite étale morphism of locally noetherian schemes, the injection $Y \xrightarrow{id \times id} Y \times_S Y$ is an open and closed immersion.*

We may now show the following, an analogue to Lemma 2.2.7:

Lemma 4.1.3. *For $(Y \xrightarrow{f} S)$ an étale covering, any section $s \to Y$ of f is an open and closed immersion.*

Proof. Using the lemmata above, this follows directly from Lemma 2.1.1, as in 2.2.7. ∎

Lemma 4.1.4. *Let $(Y \xrightarrow{f} S)$ and $(X \xrightarrow{g} S)$ be étale coverings. Then any section $s : Y \to Y \times_S X$ is an open and closed immersion.*

Proof. Using Lemma 4.1.1, we can simply invoke Lemma 4.1.3, and we are done. ∎

Keeping S fixed, we can examine the category ${}^{Et}/_{S}$ and the properties it exhibits.

Theorem 4.1.5. *The category ${}^{Et}/_{S}$ exhibits the following properties:*

1. *$S \xrightarrow{1_S} S$ is a terminal object of ${}^{Et}/_{S}$*
2. *$(\varnothing, f_\varnothing)$ constitutes an initial element of ${}^{Et}/_{S}$.*
3. *For any two objects $(X \xrightarrow{f_X} S)$, $(Y \xrightarrow{f_Y} S) \in Ob({}^{Et}/_{S})$, $(X \amalg Y \xrightarrow{f_{X \amalg Y}} S)$ is also an element of $Ob({}^{Et}/_{S})$, with $f_{X \amalg Y}$ defined in the obvious way.*
4. *The fiber product of finitely many objects $\{(X_i \xrightarrow{f_{X_i}} S)\}$ is again an object on ${}^{Et}/_{S}$.*
5. *A morphism of objects $f : X \to Y$ in ${}^{Et}/_{S}$ can be factored into a pair of morphisms $X \overset{f_1}{\twoheadrightarrow} Y_1 \overset{f_2}{\hookrightarrow} Y$, where f_1 is an effective epimorphism, f_2 is a monomorphism, and both Y_1 and Y_2 are objects of ${}^{Et}/_{S}$ for $Y = Y_1 \amalg Y_2$.*

Proof.

1. $S \xrightarrow{1_S} S$ is trivially an étale covering, and as for any given $X \xrightarrow{f_X} S$, there exists only one map $f : X \to S$ such that $1_S \circ f = f_X$ (which is, of course, f_X itself), $S \xrightarrow{1_S} S$ is a terminal object of ${}^{Et}/_{S}$
2. Likewise, there exists a trivial étale mapping $f_\varnothing : \varnothing \to S$ sending nothing nowhere, and as such, $\varnothing \xrightarrow{f_\varnothing} S \in \mathrm{Ob}({}^{Et}/_{S})$. But, as there is a unique morphism $f_{\varnothing,X} : \varnothing \to \varnothing \subseteq X$ for any $(X \xrightarrow{f_X} S) \in \mathrm{Ob}({}^{Et}/_{S})$ such that $f_\varnothing = f_X \circ f_{\varnothing,X}$, degenerate though it may be, $(\varnothing, f_\varnothing)$ constitutes an initial element of ${}^{Et}/_{S}$.
3. As the properties specifying an étale mapping are local in both the source and target schemes, this mapping is still étale. It remains finite because the product of any two finitely-generated modules is also a finitely-generated module, and so by Lemma 3.2.4 and Theorem 3.3.8, the preimages of any affine cover of Y exhibit the necessary properties.

4. Because we only require that finite fiber products exist, we may reduce to the pairwise case. For objects $(X \xrightarrow{f_X} S)$ and $(Y \xrightarrow{f_Y} S)$, we wish first to show that $X \underset{S}{\times} Y$ is well-defined as a scheme. It is certainly well-defined as a set, which we must now overlay with a well-defined sheaf of rings. We ascribe to it a basis of open sets given by $f^{-1}(U) \underset{U}{\times} g^{-1}(U) \bigcap X \underset{S}{\times} Y$ for any affine subscheme U in S, which we know to be affine by lemma 3.3.5. This provides both a topology and a $\mathfrak{B}$-sheaf of rings, well-defining a sheaf. Because the tensor product is again Noetherian, we have well-defined the fiber product as a locally Noetherian scheme. The tensor product of two finitely-generated modules is again finitely generated, and if the modules satisfy the étale property, then the tensor product again takes the form ${}^{\mathcal{O}_S(U)[x]}/_{< h >}$, with h' invertible in ${}^{\mathcal{O}_S(U)[x]}/_{< h >}$.

5. We first set Y_1 to be the image of f in Y. We then consider the following pullback:

$$\begin{tikzcd}
Y \arrow[d, "f_Y"'] & Y \underset{S}{\times} X \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
S & X \arrow[l, "f_X"] \arrow[ul, "f"'] \arrow[u, bend right=80, "f \times id"']
\end{tikzcd}$$

Because $p_2 \circ (f \times id) = id$ on X, $(f \times id)$ is a section and therefore an open and closed immersion. The projection map p_1 is a closed map, and so the image of f in Y is closed. We already know, however, that this image is Y_1, an open subscheme of Y, and so Y_1 is open and closed in Y, making $Y = Y_1 \amalg Y_2$, both open subschemes, and therefore objects of ${}^{Et}/_{S}$ by Theorem 3.3.8 and Corollary 3.3.11. As Y_1 is the image of f, f is epimorphic onto Y_1, and the inclusion of Y_1 into Y is clearly monomorphic.

■

Definition 64. For S a connected locally Noetherian scheme, we define the *fiber functor* over a geometric point s in S to be the functor $\mathcal{F}_{Et/S,s}$: ${}^{Et}/_{S} \to \mathfrak{Set}$, which associates to an étale covering $X \xrightarrow{f} S$ of S the set of geometric points in X with value in the separable completion $\overline{\mathfrak{K}(s)}$ of $\mathfrak{K}(s)$ which map to s under f. (Or, more simply, the set $f^{-1}(s)$.) We denote these associations by $\mathcal{F}_{Et/S,s}$: $X \mapsto \mathcal{F}_{Et/S,s}(X)$, and $g \mapsto \mathcal{F}_{Et/S,s}(g)$, with $\mathcal{F}_{Et/S,s}(g) : \mathcal{F}_{Et/S,s}(X) \to \mathcal{F}_{Et/S,s}(Y)$ for any morphism of objects $g : X \to Y$, such that the following diagram commutes:

Definition 65. A *pointed object* (X, x) of ${}^{Et}/_{S}$ is an object $X \xrightarrow{f} S$ in $Ob({}^{Et}/_{S})$ paired with a point x in $\mathcal{F}_{Et/S,s}(X)$ for a specified point s in S. Note that we may also simply consider (X, x) the object $X \xrightarrow{f} S$ paired with a geometric point x in X, which then specifies the fiber we are to consider as $f^{-1}(f(x))$. Be aware that the concept of a *pointed object* has a more precise and generalizable Category-Theoretic definition, which in this case would emphasize the role of x as a morphism of schemes $Spec(\overline{\mathfrak{K}(s)}) \to X$. Either emphasis is correct to be used as useful.

Definition 66. A *morphism of pointed objects* $(X, x) \to (Y, y)$ in $Ob({}^{Et}/_S)$ is simply a morphism of objects $X \to Y$ such that $x \mapsto y$.

Definition 67. An object $X \xrightarrow{f} S$ in $Ob({}^{Et}/_S)$ is called *connected* if it cannot be decomposed into $X_1 \amalg X_2$ for any pair of objects X_1, X_2 in $Ob({}^{Et}/_S)$. We note that as open subsets are open subschemes, connected objects are necessarily exactly those connected in the topological sense as well.

Lemma 4.1.6. *The fiber $f^{-1}(s)$ of any étale covering $(X \xrightarrow{f} S)$ over a point s in S is a finite set.*

Proof. Suppose not. Then there exists some affine neighborhood U containing s whose preimage consists of infinitely many disjoint affine subschemes in S. This in turn would mean that the ring associated to the preimage of this neighborhood would be the product of infinitely many finitely-generated $U-$modules, which would no longer be finitely generated, rendering f not a finite map. ■

Lemma 4.1.7. *If $(X \xrightarrow{f} S)$ is a connected object of ${}^{Et}/_S$, then any element u of $Hom_{{}^{Et}/_S}(X, X)$ (the set of morphisms of objects in ${}^{Et}/_S$ from X to itself) is an automorphism of X over S.*

Proof. We have specified both S and X to be connected. As X is connected and can only be decomposed into $X \amalg \varnothing$, by Theorem 4.1.5, we know that u is an effective epimorphism, and so the morphism of fibers (sets) $\mathcal{F}_{{}^{Et}/_S,s}(u) : \mathcal{F}_{{}^{Et}/_S,s}(X) \to \mathcal{F}_{{}^{Et}/_S,s}(X)$ is a surjective map from a finite set to itself, which must therefore be bijective. As this is true for all s in S, we conclude that u is bijective and therefore an automorphism. ■

Lemma 4.1.8. *Let (X, x), (Y, y) be a pair of pointed objects in ${}^{Et}/_S$ with X connected. Then if there exists a morphism of pointed objects $u : (X, x) \to (Y, y)$, it is unique.*

Proof. By Theorem 4.1.5, if X is connected, the image of u is epimorphic onto a single connected component of Y, and so we can reduce to the case in which both X and Y are connected objects, where we take Y to be the connected component of the target containing y. Let u, u' be two morphisms $(X, x) \to (Y, y)$. We now examine the following pullback:

$$\begin{tikzcd}
Y \arrow[d, "f_Y"'] & Y \underset{S}{\times} Y \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
S & Y \arrow[l, "f_Y"'] \arrow[u, bend right=60, "id \times id"']
\end{tikzcd}$$

As before, $p_2 \circ (id \times id) = id$ on Y, and is therefore an open and closed immersion. We use this fact in the following pullback diagram:

$$\begin{tikzcd}
Y \arrow[d, "id \times id"'] & Y \underset{Y \underset{S}{\times} Y}{\times} X \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
Y \underset{S}{\times} Y & X \arrow[l, "u \times u'"']
\end{tikzcd}$$

As open and closed immersions are preserved by pullback, p_2 is also an open and closed immersion. As $Y \underset{Y \underset{S}{\times} Y}{\times} X$ amounts to the points x' such that $u(x') = u'(x')$, and X is connected, we have that if u and u' agree on any point x' in x, then they are equal across all of X and therefore equal exactly. ∎

Corollary 4.1.9. *For $(X \xrightarrow{f} S)$ a connected object of ${}^{Et}/_{S}$, the automorphism group $Aut({}^{X}/_{S})$ acts freely on the fiber $\mathcal{F}_{{}^{Et}/_{S},s}(X)$ and is finite.*

Proof. For x, x' elements of $\mathcal{F}_{{}^{Et}/_{S},s}(X)$, there exists at most one morphism between the pointed objects $(X,x) \to (X,x')$. For $x = x'$, we get that only the identity in $Aut({}^{X}/_{S})$ fixes any element x, making the action free. Only a finite group can act freely on a finite set, and so we are done. ∎

Within the category ${}^{Et}/_{S}$, there are objects whose properties and relevance to the construction of a fundamental group bear direct analogy to Galois field extensions. Much as in the topological case, we call these *Galois objects* by way of analogy.

Definition 68. An object $(X \xrightarrow{f} S)$ of ${}^{Et}/_{S}$ is called a *Galois object* if it is connected and $Aut({}^{X}/_{S})$ acts transitively on the fiber $\mathcal{F}_{{}^{Et}/_{S},s}(X)$ for every s in S.

We note that this property, along with a specified point x in each fiber of X over S, specifies an isomorphism of $Aut({}^{X}/_{S})$−sets between each fiber and $Aut({}^{X}/_{S})$ itself.

Lemma 4.1.10. *An object $(X \xrightarrow{f} S)$ of ${}^{Et}/_{S}$ is Galois if and only if the fiber product $X \underset{S}{\times} X$ is isomorphic to the disjoint union of a set of copies of X.*

Proof. We begin by designating the size of the fiber in X over each s in S as n. It then follows that the size of the fiber in $X \times_S S$ over each s in S is n^2. We then examine the following pullback:

$$\begin{tikzcd}
\gamma X \arrow[d, "f_X"'] \arrow[r, bend left=60, "id\times\gamma"] & X \times_S X \arrow[l, "p_2"'] \arrow[d, "p_1"] \\
S & \gamma' X, \arrow[l, "f_X"] \arrow[ul, "\gamma'\circ\gamma^{-1}"'] \arrow[u, bend right=60, "\gamma'\times id"']
\end{tikzcd}$$

where γ and γ' are automorphisms in $Aut({}^{X}/_{S})$

We now note that $p_1 \circ (\gamma' \times id) = id$ on X, as does $p_2 \circ (id \times \gamma)$, meaning that both of these are sections and therefore open and closed immersions. Because $p_2 \circ (\gamma' \times id) = \gamma'$, an automorphism, we then see that any automorphism factors through $X \times_S X$ in this way, with X mapping surjectively onto an open and closed component of $X \times_S X$ under $(\gamma' \times id)$, and so the image of X under $(\gamma' \times id)$ is isomorphic to X. However, any subset of $X \times_S X$ isomorphic to X must necessarily come equipped with an isomorphism from X, and likewise, from γX, and so such a map must necessarily be able to be put as $X \xrightarrow{(\gamma',\gamma)} X \times_S X$.

If X is a Galois object, there exist n^2 such pairings (γ', γ), and so such isomorphisms cover all of $X \times_S X$. If not, then by Lemmata 4.1.7 and 4.1.8, fewer than n automorphisms of X exist and such isomorphisms cannot cover all of $X \times_S X$. Therefore, the condition is both necessary and sufficient. ∎

Lemma 4.1.11. *For $(X \xrightarrow{f_X} S)$, $(Y \xrightarrow{f_Y} S)$, and $(Z \xrightarrow{f_Z} S)$ connected objects of ${}^{Et}/_S$, with Y Galois, then for any two morphisms of objects $g_1, g_2 : X \to Y$, there exists a unique element φ of $Aut({}^Y/_S)$ such that $g_2 = \varphi \circ g_1$, and for any two morphisms of objects $h_1, h_2 : Y \to Z$, there exists a unique element ς of $Aut({}^Y/_S)$ such that $h_2 = h_1 \circ \varsigma$.*

Proof. We first designate x in X and y, y' in Y such that $f_X(x) = s$, $g_1(x) = y$, and $g_2(x) = y'$. Then, because Y is Galois, there exists some unique φ such that $\varphi(y) = y'$. Then $\varphi \circ g_1$ is a morphism of pointed objects $(X, x) \to (Y, y')$, as is g_2. By Lemma 4.1.8, they must be the same.

Lemma 4.1.8 tells us that if there exists an automorphism ς in $Aut({}^Y/_S)$ such that $h_2 = h_1 \circ \varsigma$, it is unique. We know from Theorem 4.1.5 that h_1 and h_2 are epimorphisms, so for a given z in the image of h_1 in Z, there exist some y, y' such that $h_1(y') = h_2(y) = z$. Then, we know there exists a unique automorphism ς sending y to y', and so $h_1 \circ \varsigma$ is a morphism sending y to $h_2(y)$, which must uniquely be h_2. ∎

This shows that if a morphism between $Y \to Z$ as given above exists, the automorphisms of Y uniquely determine those of Z, as in the Galois case in Topology or Galois Theory. However, the construction of the étale fundamental group relies on the existence of a system of Galois objects which so surject over every object $(Z \xrightarrow{f_Z} S)$ in ${}^{Et}/_S$. Such a system must always exist, but its existence is not obvious.

Definition 69. A *Galois closure* of a connected object $(X \xrightarrow{f_X} S)$ in ${}^{Et}/_S$ is a Galois object $(Y \xrightarrow{f_Y} S)$ together with a morphism of objects $g : Y \to X$ such that for every Galois object $(Z \xrightarrow{f_Z} S)$ with a morphism $h : Z \to X$, h factors through Y.

Theorem 4.1.12. *Any connected object $(Z \xrightarrow{f_Z} S)$ in ${}^{Et}/_S$ has a Galois closure $(X \xrightarrow{f_X} S)$, unique up to isomorphism.*

Proof. This proof is reproduced and expanded upon from [Mézard], wherein it is Lemma 2.10. Suppose the fiber in X over some point s in S is $f_X^{-1}(s) = \{x_1, \dots, x_n\}$. Then we consider the fiber product over S of n copies of X, $X_1 \underset{S}{\times} \cdots \underset{S}{\times} X_n$, which we denote X^n (This is not the same as $X_1 \times \cdots \times X_n$!). Specifically, we consider the connected component containing the ordered $n-$tuple $(x_1, \dots, x_n)$, which we will for convenience denote ξ. We call this component Y, and claim that it satisfies all criteria to be the Galois closure over X.

We first show it is Galois. Let us denote for every i, j in $\{1, \dots, n\}$ the function $p_{i,j} : X^n \to X \underset{S}{\times} X$ to be the projection in the i^{th} and j^{th} onto $X \underset{S}{\times} X$. We denote Δ' to be the diagonal of $X \underset{S}{\times} X$, and define $\Delta = \bigcup_{i,j \in \{1,\dots,n\}, i<j} p_{i,j}^{-1}(\Delta')$. Because Y is

connected, unless $Y \bigcap \Delta = \varnothing$, Y must be contained entirely within Δ. But as no two coordinates of ξ project to the same point, $p_{i,j}(\xi)$ is not in Δ' for any i, j, so $Y \bigcap \Delta = \varnothing$.

Thus, every element of Y has distinct coordinates, and so we can write any element of the fiber over s in Y as $\eta = (x_{j_1}, \ldots, x_{j_n})$. As there exists a unique σ element of $\mathcal{S}_n$ (the permutation group on n letters) sending $(1, \ldots, n)$ to $(\sigma(1), \ldots, \sigma(n)) = (j_1 \ldots, j_n)$, we can identify any element of $Aut({}^{Y}\!/S)$ with a corresponding element in $\mathcal{S}_n$, making $Aut({}^{Y}\!/S)$ isomorphic to some subset of $\mathcal{S}_n$. As any morphism $\omega : Y \to X$ must be epimorphic with X connected, we find that $\mathcal{F}_{Et/S,s}(\omega)$ is a surjection of sets, and so for every i, there exists some element $\eta = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ in $\mathcal{F}_{Et/S,s}(Y)$ (for some permutation σ). We may now consider the action of the permutation σ on X^n, wherein the symmetry of X^n in coordinates makes σ clearly an automorphism, and specifically, the action of σ on Y as a subset of X^n. Because Y is connected, the image $\sigma(Y)$ must be connected, and as η is in $\mathcal{F}_{Et/S,s}(\sigma(Y))$ as well as in $\mathcal{F}_{Et/S,s}(Y)$, the two sets must coincide entirely, and therefore σ is an automorphism of Y. But as this argument applies to any η in $\mathcal{F}_{Et/S,s}(Y)$, ξ is in the same orbit as every other element, and so the action of $Aut({}^{Y}\!/S)$ is transitive. Thus, Y is Galois, and we have already demonstrated it has an epimorphism onto X.

What remains to be shown are the factoring property and uniqueness. We now let $(Z \xrightarrow{f_Z} S)$ be another Galois object with $Z \xrightarrow{v} X$ a morphism onto X (necessarily an epimorphism because X is connected). Because this is epimorphic, we know there exist for all i some η_i in $\mathcal{F}_{Et/S,s}(X)$ such that the induced map $\mathcal{F}_{Et/S,s}(v) : \mathcal{F}_{Et/S,s}(Z) \to \mathcal{F}_{Et/S,s}(X)$ sends η_i to x_i. By Lemma 4.1.11, we know there exists some unique automorphism ϱ_i in $Aut({}^{Z}\!/S)$ such that $\mathcal{F}_{Et/S,s}(\varrho_i)(\eta_1) = \eta_i$.

We now construct $\gamma = \prod_{i=1}^{n} v \circ \varrho_i : Z \to X^n$. Now, $\gamma(\eta_1) = \xi$ in Y, so we know the image of γ is Y. Moreover, we see that any map $Z \to X$ is the composition of $p_1 \circ v$ with an automorphism, which makes it factor through Y by Lemma 4.1.11.

The application of this property to any other Galois closure of X yields uniqueness up to isomorphism directly. ■

Lemma 4.1.13. *For any object $X \xrightarrow{f_X} S$ of ${}^{Et}\!/S)$ and any two points s, s' in X, the fibers $f^{-1}(s)$ and $f^{-1}(s')$ have the same number of elements, and are isomorphic as $Aut({}^{X}\!/S)$−sets.*

Proof. Because any object of ${}^{Et}\!/S$ is the disjoint union of connected objects, we can take X to be connected. We then consider $f : P \to X$ to be a map from P, the Galois closure of X. As every morphism $P \to P$ is an automorphism over S, the automorphisms of P over X are exactly those automorphisms of P over S which preserve the fibers of X. As $Aut({}^{P}\!/S)$ acts freely and transitively on the fibers over s in P, every fiber $f_P^{-1}(s)$ is the same size, and $Aut({}^{P}\!/S)$ acts transitively on $f_P^{-1}(S)$ and therefore $f^{-1}(x)$. Granted, some elements of $Aut({}^{P}\!/S)$ may (so far as we know at this point), send an element of $f^{-1}(x)$ to a different fiber. We still know, though, that some subgroup of $Aut({}^{P}\!/S)$ acts transitively on $f^{-1}(x)$. However, by Lemma 4.1.11, we can describe any element of $Aut({}^{X}\!/S)$ by a subgroup of $Aut({}^{P}\!/S)$, and $Aut({}^{X}\!/S)$

itself therefore as a subgroup of $Aut({}^{P}/_{S})$. Since $Aut({}^{X}/_{S})$ acts freely on the fiber $f_X^{-1}(s)$ each action on $f^{-1}(x)$ has the same stabilizer conjugacy class, and therefore, the sizes of $f^{-1}(x)$ are the same, so the sizes of each $f_X^{-1}(s)$ must be the same also, with that action and the choice of a point in each fiber forcing an isomorphism of $Aut({}^{X}/_{S})-$sets between them. ■

4.2 The Étale Fundamental Group

With all the pieces in place, we may finally define the étale fundamental group.

Definition 70. The *étale fundamental group* $\pi_1(S, s)$ at a geometric point s of a connected, locally Noetherian scheme S is the group of automorphisms of the fiber functor $\mathcal{F}_{Et/S,s} : {}^{Et}/_{S} \to \mathfrak{Set}$, acting on the right. This is to say, $\pi_1(S, s)$ is the group of natural transformations from the fiber functor $\mathcal{F}_{Et/S,s}$ to itself. An element of $\pi_1(S, s)$ is a collection of automorphisms $\{\phi_X\}$, with ϕ_X in $Aut(\mathcal{F}_{Et/S,s}({}^{X}/_{S}))$ for all objects X of ${}^{Et}/_{S}$, which commute with pointed maps of covering spaces.

Theorem 4.2.1. *Let $\{P_i\}$ be a collection of Galois objects of ${}^{Et}/_{S}$ such that for all connected objects X in ${}^{Et}/_{S}$, there exists some epimorphism $P_i \to X$ for some i (in which case, we say P_i* trivializes *X and $\{P_i\}$ is a* cofinal system *of Galois objects). Then for any s in S, $\pi_1(S, s) \cong \varprojlim_i Aut({}^{P_i}/_{S})$. In particular, this is true when $\{P_i\}$ ranges over all Galois objects.*

Proof. We begin by noting that Theorem 4.1.12 guarantees the existence of such a system. We then note that by Lemma 4.1.11, if P is a Galois object which trivializes an object of X, then any automorphism of X is completely determined by a (not generally unique) automorphism of P. Thus, any collection $\{\phi_X\}$ of automorphisms which commute with $\mathcal{F}_{Et/S,s}$ is uniquely determined by the subcollection $\{\phi_{P_i}\}$. We can then identify $\pi_1(S, s)$ as the group of collections of automorphisms $\{P_i\}$ which commute with pointed maps between them. But because the objects of $\{P_i\}$ also trivialize other P_i, we are given a collection of surjective homomorphisms of groups $Aut({}^{P_i}/_{S}) \to Aut({}^{P_j}/_{S})$ supplied by the existence of a morphism $P_i \to P_j$. Therefore, $\pi_1(S, s)$ is the set of elements of $\prod_i Aut({}^{P_i}/_{S})$ which commute with the homomorphisms given, which is the definition of $\varprojlim_i Aut({}^{P_i}/_{S})$. ■

This construction shows $\pi_1(S, s)$ to be a profinite group, equal to its own profinite completion. The reader may note the similarity between the construction of $\pi_1(S, s)$ and the groups $Aut(\mathcal{F}_X^{fin})$ in the topological case and $Gal({}^{\Omega}/_{\mathbb{F}})$, the absolute Galois group of a field $\mathbb{F}$. To fully establish this similarity, however, we will need to establish a few more properties of $\pi_1(S, s)$.

Corollary 4.2.2. *For P a Galois object of ${}^{Et}/_{S}$, $Aut({}^{P}/_{S})$ is a finite quotient group of $\pi_1(S, s)$.*

Proof. We can use the system of all Galois objects as the cofinal system described in Theorem 4.2.1. This construction gives $\pi_1(S,s)$ the structure of a profinite group, constructed out of a system including $Aut({}^P\!/\!_S)$, and therefore $Aut({}^P\!/\!_S)$ is a finite quotient of $\pi_1(S,s)$. ■

Lemma 4.2.3. *An object $X \xrightarrow{f_X} S$ of ${}^{Et}\!/\!_S$ is connected if and only if $\pi_1(S,s)$ acts on $\mathcal{F}_{Et/S,s}(X)$ transitively.*

Proof. As above, for P the Galois closure of a connected X, $Aut({}^P\!/\!_S)$ acts transitively on $\mathcal{F}_{Et/S,s}(X)$, and by construction, $\pi_1(S,s)$ contains $Aut({}^P\!/\!_S)$ as a finite quotient group. Now, we assume $X = X_1 \amalg X_2$ is not connected, but X_1 and X_2 are. Then the action of $\pi_1(S,s)$ is mediated through the automorphism groups $Aut({}^{P_1}\!/\!_S)$ and $Aut({}^{P_2}\!/\!_S)$ of objects P_1 and P_2, the respective Galois closures of X_1 and X_2. (As X is disconnected, it cannot have a Galois closure, as the image of a connected component under morphism must be another connected component). Thus, $\pi_1(S,s)$ acts by $Aut({}^{P_1}\!/\!_S) \times Aut({}^{P_2}\!/\!_S)$, which does not transpose elements of X_1 with those of X_2. Thus, the action is not transitive.

Finally, any disconnected X, $X = X_1 \amalg X_2 \amalg X'$, with X_1 and X_2 connected and X' some other object. Thus, the proof holds for the general disconnected X. ■

Lemma 4.2.4. *For a connected, nonempty object $X \xrightarrow{f_X} S$ of ${}^{Et}\!/\!_S$ and $N \lhd \pi_1(S,s)$ the kernel of the action of $\pi_1(S,s)$ on $\mathcal{F}_{Et/S,s}(X)$, X is Galois if and only if ${}^{\pi_1(S,s)}\!/\!_N$ acts freely and transitively on X.*

Proof. By Corollary 4.2.2, we can see that if X is Galois, $Aut({}^P\!/\!_S)$ is a finite quotient group of $\pi_1(S,s)$. As an element of $\pi_1(S,s)$ is simply a collection of automorphisms, and $Aut({}^P\!/\!_S) \cong {}^{\pi_1(S,s)}\!/\!_N$, for N some normal subgroup of $\pi_1(S,s)$, we can see that N consists exactly of those elements of $\pi_1(S,s)$ for which φ_X is the identity. This necessarily equates N with the kernel of the action on $\mathcal{F}_{Et/S,s}(X)$. As $Aut({}^P\!/\!_S)$ acts freely and transitively, we are done.

We now take X to be not Galois, and consider the action of ${}^{\pi_1(S,s)}\!/\!_N$. As X is connected, we know it must act transitively. If it acts freely, then there is a set of automorphisms of the Galois closure P of X which acts freely and transitively on $\mathcal{F}_{Et/S,s}(X)$. If the action is free, then each must restrict to a different automorphism of X, and so the action of $Aut({}^X\!/\!_S)$ on $\mathcal{F}_{Et/S,s}(X)$ must also be free and transitive, which contradicts our assumption that X was not Galois, and so we are done. ■

The following corollary is an immediate consequence:

Corollary 4.2.5. *For $X \xrightarrow{f_X} S$ a nonempty Galois object of ${}^{Et}\!/\!_S$, $Aut({}^X\!/\!_S) \cong {}^{\pi_1(S,s)}\!/\!_N$, for N the kernel of the action of $\pi_1(S,s)$ on X, equivalent to taking N the stabilizer of any element of $\mathcal{F}_{Et/S,s}(X)$.*

Lemma 4.2.6. *For $X \xrightarrow{f_X} S$ and $Y \xrightarrow{f_Y} S$ objects of ${}^{Et}\!/\!_S$, morphisms of objects $X \to Y$ bijectively correspond to morphisms of $\pi_1(S,s)-$sets between $\mathcal{F}_{Et/S,s}(X) \to \mathcal{F}_{Et/S,s}(Y)$.*

Proof. Any morphism $X \to Y$ must send every x in X to some y in Y, and because the morphism must commute with the maps onto S, $f_X(x) = f_Y(y) = s$. And because elements of $\pi_1(S,s)$ must commute with such morphisms, the structure of the $\pi_1(S,s)-$sets is preserved. Thus, any morphism $X \to Y$ clearly induces a morphism of $\pi_1(S,s)-$sets $\mathcal{F}_{Et/S,s}(X) \to \mathcal{F}_{Et/S,s}(Y)$.

Because every morphism q can be broken down into its mapping from each connected component of X to some connected component of Y, we may reduce to the case where X and Y are connected, wherein $\mathcal{F}_{Et/S,s}(X)$ and $\mathcal{F}_{Et/S,s}(Y)$ each become a single $\pi_1(S,s)-$orbit. We now suppose we have a morphism of $\pi_1(S,s)-$sets $q : \mathcal{F}_{Et/S,s}(X) \to \mathcal{F}_{Et/S,s}(Y)$. By Lemma 4.1.8, we know that if any morphism of objects of ${}^{Et}/_S$ $\tilde{q} : X \to Y$ induces q, it is unique. We now need only show that for every such $\tilde{q}$, some such q induces it. Such a morphism of $\pi_1(S,s)-$sets is, by definition, a function $\tilde{q}$ such that $\tilde{q}(g \cdot x) = g \cdot \tilde{q}(x)$, for all g in $\pi_1(S,s)$ and all x in $\mathcal{F}_{Et/S,s}(X)$.

Now, for a Galois object to trivialize an object, we need only know that the kernel of the $\pi_1(S,s)-$action on that object contains the kernel of the $\pi_1(S,s)-$action on the Galois object. In order for there to exist such a morphism q, mapping x to y, the stabilizer of each x must be contained within the stabilizer of its image $q(x)$. This means that for P the Galois closure of X, P also trivializes Y. This trivialization means that there exist points p, p' and maps $\rho_X : P \to X$ and $\rho_Y : P \to Y$ such that $\rho_X : p \mapsto x$ and $\rho_Y : p \mapsto y$, as well as an automorphism γ sending p to p'. That the stabilizer of x is contained within the stabilizer of y means that any element of $\rho_Y^{-1}(y)$ is contained within $\gamma(\rho_X^{-1}(x))$, and so there is a well-defined map sending x to $\rho_Y(\gamma(\rho_X^{-1}(x)))$ which commutes with the mappings onto S, and is therefore a morphism of objects. ∎

We now venture beyond the scope of this paper for a moment to list a few properties of the étale fundamental group important to its further study, deferring to [Mézard], section 2.15 for further discussion. The first is that, while we can think of $\pi_1(S,s)$ as the projective limit of the automorphism groups of a system of Galois objects, so constructing it as a group out of groups, we can also think of it as the automorphism group of the projective limit of that same system. This is to say as follows:

Theorem 4.2.7. *For $\{P_i\}$ a cofinal system of Galois objects of ${}^{Et}/_S$ partially ordered by the existence of a morphism of objects $P_i \to P_j$, there exists a scheme $P = \varprojlim_i P_i$, unique independent of choice of $\{P_i\}$, equipped with a map $f_P : P \to S$ and a map $f_\alpha : P \to X_\alpha$ for every object X_α of ${}^{Et}/_S$ which commutes with all morphisms of objects and all covering maps onto S. The fiber functor acts on this scheme P such that $\mathcal{F}_{Et/S,s}(P) = \varprojlim_i(\mathcal{F}_{Et/S,s}(P_i)$. The group of automorphisms of P over S is exactly $\pi_1(S,s)$.*

It is worth pointing out that P is very rarely an object of ${}^{Et}/_S$, as the morphism onto S is not generally finite.

The next result establishes a relationship between fiber functors over different geometric points, justifying the association of the étale fundamental group to a space, rather than a single point.

Theorem 4.2.8. *For a connected, locally Noetherian scheme S containing distinct geometric points s and s', there is an isomorphism defined up to inner automorphism between $\pi_1(S,s) \cong \pi_1(S,s')$.*

The final outside theorem solidifies the relationship between $\pi_1^{top}(X,x)$ and $\pi_1(S,s)$ beyond analogy. It requires one definition, however.

Definition 71. A morphism of schemes $f : X \to Y$ is called *of finite type* if for every point y in Y, there exists an affine open neighborhood U_i of Y containing y such that there exists a finite affine cover $\{V_{i,j}\}$ of $f^{-1}(U_i)$ where the restriction of f to $V_{i,j} \to U_i$ induces a map of rings $\mathcal{O}_Y(U_i) \to \mathcal{O}_X(V_{i,j})$ which gives $\mathcal{O}_X(V_{i,j})$ the structure of a finitely-generated $\mathcal{O}_Y(U_i)-$algebra. In this case, we say X is *of finite type over Y*.

Theorem 4.2.9. *(Riemann Existence Theorem): Let X be a scheme of finite type over $\mathbb{C}$. There is an equivalence of categories between finite étale coverings of X and finite topological coverings of $X(\mathbb{C})$.*

While the proof of this supposition is beyond our scope, we recognize its elegance and importance, and so direct the curious reader to [Hartshorne], wherein it is discussed in Theorems 3.1 and 3.2.

The following corollary follows directly from the equivalence of categories (and that it induces an equivalence of automorphism groups) and Theorem 2.2.2.

Corollary 4.2.10. *For X as above, for any x in X and for any c in $X(\mathbb{C})$,*
$\pi_1(X,x) \cong \pi_1^{top}\widehat{(X(\mathbb{C})}, c)$.

4.3 Computation of an Étale Fundamental Group

As we have just solidified the connection between finite topological coverings, étale coverings, and the fundamental groups of each, it now falls to us to connect our discussion of Galois Theory beyond mere analogy. As such, we now take it upon ourselves to compute the étale fundamental group of the scheme $Spec(\mathfrak{K})$, where $\mathfrak{K}$ is a field.

We begin with a discussion of étale coverings of $\mathfrak{K}$. To begin, we know from Theorem 3.3.1 that, as $Spec(\mathfrak{K})$ is affine, maps from another scheme $(X, \mathcal{O}_X)$ will be in bijective correspondence to homomorphisms of rings $\mathfrak{K} \to \mathcal{O}_X(X)$. We know that all étale coverings are affine morphisms, and as such, Theorem 3.3.7 tells us that all étale coverings of an affine scheme must have a source scheme that is affine also. If we take X to be nonempty, we know that $\mathfrak{K}$ must be isomorphic to its image in $\mathcal{O}_X(X)$. We know from Theorem 3.3.8 that X must have a covering by open affines U_α such that $\mathcal{O}_X(U_\alpha)$ must be a finite $\mathfrak{K}-$module. We may now quote the Galois Theory result Lemma 2.3.3 to see that each of these U_α must be isomorphic to $\mathfrak{K}[\alpha]$, for some α algebraic over $\mathfrak{K}$. As each such U_α would then have to consist of a single point, we see that any two U_α cannot be glued together along an open subset without being completely identified together, and so X must be the disjoint union of finitely many such $Spec(\mathfrak{K}[\alpha])$. This gives us the following lemma:

Lemma 4.3.1. *$X \xrightarrow{f_X} Spec(\mathfrak{K})$ is a connected étale covering if and only if $Y \cong Spec(\mathfrak{L})$, and f_Y is induced by an injection $\mathfrak{K} \hookrightarrow \mathfrak{L}$, a finite separable extension of $\mathfrak{K}$.*

We now consider automorphisms of an object $X \xrightarrow{f_X} Spec(\mathfrak{K})$, where $X \cong Spec(\mathfrak{L})$. As a map $X \to X$ is a morphism of affine schemes, we see that it must be associated uniquely to a map $\mathfrak{L} \hookrightarrow \mathfrak{L}$, and as it must commute with the map f_X, it must preserve the injection $\mathfrak{K} \hookrightarrow \mathfrak{L}$. Such a map is exactly an automorphism of $\mathfrak{L}$ over $\mathfrak{K}$, and so we get the following:

Lemma 4.3.2. *For $Spec(\mathfrak{L}) \to Spec(\mathfrak{K})$ an étale covering, the groups $Aut({}^{Spec(\mathfrak{L})}/_{Spec(\mathfrak{K})})$ and $Gal({}^{\mathfrak{L}}/_{\mathfrak{K}})$ are isomorphic.*

Finally, we examine the fiber over a geometric point k in $Spec(\mathfrak{K})$ in a connected object $Spec(\mathfrak{L}) \to Spec(\mathfrak{K})$, which is simply an injection $\mathfrak{L} \to \Omega$ which preserves the image of $\mathfrak{K}$ specified by the geometric point k. The Primitive Element Theorem tells us that such an injection is uniquely determined by the image of its primitive element, which we may call $\alpha_{\mathfrak{L}}$. As $\alpha_{\mathfrak{L}}$ is algebraic over $\mathfrak{K}$, we know that it has exactly n conjugate roots in Ω, and so the fiber over k has n elements. We then know that $Spec(\mathfrak{L}) \to Spec(\mathfrak{K})$ is a Galois object of ${}^{Et}/_{Spec(\mathfrak{K})}$ if and only if $Aut({}^{Spec(\mathfrak{L})}/_{Spec(\mathfrak{K})})$ has exactly n elements, which we can say from Lemma 4.3.2, if and only if $Gal({}^{\mathfrak{L}}/_{\mathfrak{K}})$ has exactly n elements. But this only occurs when the extension $\mathfrak{L} \supseteq \mathfrak{K}$ is Galois:

Lemma 4.3.3. *For $\mathfrak{K}$ a field, the Galois objects of $Spec(\mathfrak{K})$ are exactly those elements $Spec(\mathfrak{L}) \xrightarrow{f_{\mathfrak{L}}} Spec\mathfrak{K})$, where $f_{\mathfrak{L}}$ corresponds to a Galois extension $\mathfrak{K} \hookrightarrow \mathfrak{L}$.*

We are therefore, in this instance at least, well justified in referring to Galois objects as such. Now we may show the following:

Theorem 4.3.4. *For $\mathfrak{K}$ a field and k a geometric point of $Spec(\mathfrak{K})$, $\pi_1(Spec(\mathfrak{K}), k)$ is isomorphic to the absolute Galois group of $\mathfrak{K}$.*

Proof. From here, we may use Theorem4.2.1 to assemble $\pi_1(Spec(\mathfrak{K}), k)$:
$\pi_1(Spec(\mathfrak{K}), k) \cong \varprojlim_i Aut({}^{P_i}/_{Spec(\mathfrak{K})})$, for P_i a Galois object of ${}^{Et}/_{Spec(\mathfrak{K})}$
$\cong \varprojlim_i Aut({}^{Spec(\mathfrak{L}_i)}/_{Spec(\mathfrak{K})})$, for $\mathfrak{L}_i$ varying over all Galois extension of $\mathfrak{K}$
$\cong \varprojlim_i Gal({}^{\mathfrak{L}_i}/_{\mathfrak{K}})$
$\cong Gal({}^{\Omega}/_{\mathfrak{K}})$, for Ω the separable closure of $\mathfrak{K}$. ∎

We note that this is the absolute Galois group of $\mathfrak{K}$, and that both this result and Lemmata 4.3.1 and 4.3.2 justify the notion of field extensions as coverings of a field. We may also note that $Spec\Omega$ necessarily matches the construction of the object described in Theorem 4.2.7, and that this serves to ascribe to Ω a similar role in Galois Theory as the universal covering plays in topological covering spaces.

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