The Reducibility and Dimension of Hilbert Schemes of Complex Projective Curves

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Chapter 1

Introduction

A major object of study in algebraic geometry is the sets of varieties or schemes sharing important properties. Often, these sets are themselves geometric objects. A set such as the curves of a given genus or the hypersurfaces of a given degree in a projective space is parametrized by some variety or scheme.

The most basic example is the set of surfaces of degree d in a projective space $\mathbb{P}^r$. Specifying a hypersurface of degree d is equivalent to specifying a homogeneous polynomial of degree d in $r+1$ variables. The space of all such polynomials is parametrized by the coefficients of each monomial, and turns out to be a projective space. This parameter space behaves very well, and we would hope to generalize many of its properties.

Another simple motivating example is the Grassmannian parametrizing k-planes in a vector space. The Grassmannian is a projective variety, and while its geometry is more complicated than the geometry of a projective space, it is also relatively well understood.

There are two basic approaches to defining similar parameter spaces in more generality. One is to classify schemes up to isomorphism, or in other words to parametrize abstract schemes rather than their projective or affine realizations. The resulting schemes are called moduli spaces, and the most important example is probably the moduli space $\mathcal{M}_g$ parametrizing curves of genus g.

The moduli space $\mathcal{M}_g$ fails to be universal, however, in an important sense. The second approach, which can give universal families, is to parametrize subschemes of a fixed projective space. This can take several forms, but perhaps the most important is the Hilbert scheme, which parametrizes projective schemes with the same Hilbert polynomial. Another example is the Hurwitz scheme parametrizing projective curves along with simply-branched maps to $\mathbb{P}^1$.

As we will discover, the Hilbert scheme has a very natural definition. It also coincides with the projective space of polynomials for hypersurfaces and the Grassmannian for k-planes. The major question of this thesis is what Hilbert schemes parametrizing various types of projective curves look like as a geometric objects.

Unfortunately, the geometry of the Hilbert scheme is often far more pathological than basic examples indicate. Though always connected, Hilbert schemes need not be smooth, irreducible or even reduced. Nevertheless, determining its geometry is a fascinating problem that is tractable in many cases.

The major questions will be how many irreducible components various Hilbert schemes have and what the dimensions of these components are. Even in the projective space $\mathbb{P}^3$, these questions are a long way from solved. We discuss several major results and examples, and then describe the situation in low degree.

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All schemes will be over $\mathbb{C}$, and this text is primarily concerned with Hilbert schemes parametrizing complex projective curves. Because the Hilbert scheme appears in a wide variety of settings, however, the definition is given for subschemes of a projective space with an arbitrary fixed Hilbert polynomial. For curves, choosing a Hilbert polynomial is the same as choosing a degree and a genus.

Chapter 2 introduces Hilbert schemes of complex projective schemes, beginning with the very natural definition of the Hilbert scheme as a representable functor. After several basic examples, the construction of the Hilbert scheme is outlined. Finally, the tangent space to the Hilbert scheme is characterized as the sections of a normal sheaf.

The following two chapters focus on two different approaches to describing the geometry of Hilbert schemes of projective curves. Chapter 3 develops techniques for relating curves to the surfaces they lie on. When curves lie on quadric or cubic surfaces, the divisor classes of the curves on those surfaces can be understood. The theory of liaison relates curves that fit together to form complete intersections.

The techniques in Chapter 4 treat projective curves as abstract curves along with maps to a projective space. In high enough degree, Brill-Noether theory shows there is a component of the Hilbert scheme that includes a realization of each abstract curve. This theory also gives a derivation of the expected dimension of the Hilbert scheme, which is a lower bound on the dimension that is often achieved.

The final chapter asks when Hilbert schemes components have the expected dimension and how many components there are. The first example of a Hilbert scheme component larger than the expected dimension is computed. So is the first example of a Hilbert scheme with several components parametrizing smooth curves. To my knowledge, these computations are not carried out elsewhere in the literature.

The text is meant as an introduction to the theory of Hilbert schemes. I have attempted to prove simple facts which are only stated in other sources but initially gave me trouble. The text is not always self-contained, and references are given for those results (often technical) that are not proven. I have tried to state important theorems and point the reader in a few interesting directions.

The prerequisite for this text is familiarity with basic algebraic geometry, including basic properties of curves such as the Riemann-Roch theorem and some exposure to schemes and sheaf cohomology. The first four chapters of Hartshorne's *Algebraic Geometry* are more than enough background [1]. Eisenbud and Harris's book on *The Geometry of Schemes* should also suffice [2].

There are several other references that cover parts of the material in this text. The notes by Harris on *Curves in Projective Space* overlap with much of my third and fourth chapters, but also cover many more topics than my treatment [3]. The first chapter of *Moduli of Curves* by Harris and Morrison is another source, including a broad discussion of moduli problems [4]. Another introduction to the Hilbert scheme, in the context of deformation theory, is given by Hartshorne's book on *Deformation Theory* [5].

Chapter 2

The Hilbert Scheme: Definition and Examples

2.1 The Hilbert Scheme as a Representable Functor

In general, the Hilbert scheme parametrizes the set of schemes in a fixed projective space with a given Hilbert polynomial. While the following chapters are only concerned with the case of algebraic curves, where the Hilbert polynomial is $p(n) = dn + (1 - g)$, it is not more difficult to give the definition in more generality.

The Hilbert scheme's underlying set

$$\{X \subset \mathbb{P}^r : X \text{ has Hilbert polynomial } p(n)\}$$

is easy to describe, but giving a scheme structure on this set is more difficult. The construction of the Hilbert scheme is technical and often unwieldy, so only a brief sketch will be given. There is a much more natural characterization of the Hilbert scheme, however, as a representable functor.

This section will review the definition of a representable functor in the category of schemes before defining the Hilbert scheme. The idea is that a contravariant functor is representable if it maps each scheme R to the maps from R to some fixed scheme.

Definition 2.1.1. Given a scheme S, the functor of points $\mathrm{Hom}(-, S)$ of S is a contravariant functor from schemes to sets sending R to $\mathrm{Hom}(R, S)$. A contravariant functor

$$F : \text{Schemes} \to \text{Sets}$$

is representable if there exists a natural isomorphism of functors from F to $\mathrm{Hom}(-, S)$ for some scheme S.

Representable functors appear in a wide variety of settings such as algebraic geometry and algebraic topology. Yoneda's lemma implies that a scheme S representing a functor F is unique if it exists (up to a unique isomorphism of S and the isomorphism from F to $\mathrm{Hom}(-, S)$).

The functor of interest sends a scheme to the set of flat families over that scheme with well-behaved fibers. Here, a family refers to a subscheme $X \subset \mathbb{P}^r \times B$ along with a morphism $X \to B$ from X to the base B. The family is flat if the morphism is, and the notion of flatness is introduced in [1] and in [2]. The condition on the fibers of X over B is that each fiber must be a projective scheme, and all of the fibers must have the same Hilbert polynomial.

When the base scheme B is reduced, a family is flat if and only if the fibers over B share the same Hilbert polynomial. The flatness condition ensures that the family behaves reasonably well as the fibers vary. Fixing the Hilbert polynomial will also ensure that the related parameter space is not too large by specifying a degree and other geometric invariants.

The construction of the Hilbert scheme gives:

Theorem 2.1.2. *Fix a projective space $\mathbb{P}^r$ and a Hilbert polynomial $p(n)$. There exists a scheme $\mathcal{H}_{p(n),r}$ representing the functor $\boldsymbol{Hilb}_{p(n),r}$ sending a scheme B to the set of flat families $X \to B$, where $X \subset B \times \mathbb{P}^r$, such that the fibers have Hilbert polynomial $p(n)$. This scheme $\mathcal{H}_{p(n),r}$ is called the Hilbert scheme.*

Because $\mathbf{Hilb}_{p(n),r}$ is representable, the Hilbert scheme is called a fine moduli space. Many moduli problems do not admit fine moduli spaces, and a weaker notion of a coarse moduli space is needed. The moduli space $\mathcal{M}_g$ of smooth curves of genus g is the most notable such example.

Representability has a number of significant consequences. If $X \to B$ is any flat family, then the isomorphism of functors Ψ gives a map $B \to \mathcal{H}_{p(n),r}$ which is called $\Psi(X)$. Moreover, these maps are compatible with change of base. If $B' \to B$ is a morphism and $X \times_B B' \to B$ is the fiber product, then the diagram

$$\begin{tikzcd} B' \arrow[rr] \arrow[dr, "\Psi(X\times_B B')"'] & & B \arrow[dl, "\Psi(X)"] \\ & \mathcal{H}_{p(n),r} & \end{tikzcd}$$

commutes. The idea is that the Hilbert scheme parametrizes all schemes with Hilbert polynomial $p(n)$, and the map $\Psi(X)$ sends $b \in B$ to the point corresponding to the fiber over b in X.

Going in the opposite direction, the identity morphism $\mathcal{H}_{p(n),r} \to \mathcal{H}_{p(n),r}$ gives a universal family $\mathscr{H} \to \mathcal{H}_{p(n),r}$ over $\mathcal{H}_{p(n),r}$. The family is called universal because for each family $X \to B$, the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathscr{H} \arrow[d] \\ B \arrow[r, "\Psi(X)"'] & \mathcal{H}_{p(n),r} \end{tikzcd}$$

commutes and is a fiber product square. Each point $h \in \mathcal{H}_{p(n),r}$ can be thought of as parametrizing the fibers of the universal family over that point h.

In addition, representability will allow a simple characterization of the tangent space to the Hilbert scheme at each point. In practice, the definition of the Hilbert scheme as a representable functor and the properties derived from this definition are much more important the actual construction.

Two remarks are worth making before proceeding. First, there are a number of variations on the Hilbert scheme. The simplest replaces the projective space $\mathbb{P}^r$ with a closed subscheme $Y \subset \mathbb{P}^r$. Then the Hilbert functor sends a scheme B to the set of flat families $X \subset B \times Y$ such that the fibers over B have Hilbert polynomial $p(n)$. Other variations are introduced in [2].

Second, Hartshorne showed in his Ph.D. thesis that any Hilbert scheme parametrizing subschemes of $\mathbb{P}^r$ is connected [6]. Very roughly, his idea was to specialize the schemes parametrized by any connected family to schemes with simpler linear structure. As the following chapters will make clear, connectedness is perhaps the only geometric property possessed by arbitrary Hilbert schemes.

2.2 Basic Examples

An outline of the construction of the Hilbert scheme will be postponed until Section 2.3, after a few examples have been given.

Hypersurfaces: The first is Hilbert schemes parametrizing hypersurfaces of degree d in $\mathbb{P}^r$. In this case, the Hilbert polynomial is

$$p(t) = \binom{t+r}{r} - \binom{t+r-d}{r}.$$

Because each hypersurface is the zero set of a degree d polynomial, the hypersurfaces are parametrized by the projective space $\mathbb{P}^m$, where $m = \binom{d+r}{r} - 1$, giving the coefficients of each monomial. This projective space will be shown to be the Hilbert scheme as well.

Because the parameters are explicit, this is an useful motivating case for developing intuition about parameter spaces.

Let $x_0, ..., x_r$ be coordinates for $\mathbb{P}^r$ and $\{a_{i_0,...,i_d}\}_{i_0+...+i_d=d}$ be coordinates for $\mathbb{P}^m$. As a polynomial on $\mathbb{P}^m \times \mathbb{P}^r$,

$$\sum_{i_0+...+i_d=d} a_{i_0,...,i_d} x_0^{i_0} ... x_d^{i_d}$$

cuts out a family over $\mathbb{P}^m$. The claim is that this is the universal family corresponding to $\mathcal{H}_{p(t),r}$.

Let $X \to B$ be any flat family such that the fibers have Hilbert polynomial $p(t)$. Assume without loss of generality that B is affine. If F is a fiber over $b \in B$, then $F = F' \cup F''$, where F' is a hypersurface of degree d and F'' has dimension at most $r-2$. But then the Hilbert polynomial $p(t)$ of F is equal to the sum of the Hilbert polynomial of F' and the Hilbert polynomial of the ideal sheaf $\mathcal{I}_{F'/F}$. Because the Hilbert polynomial of F' is $p(t)$, the conclusion is $F = F'$. Each fiber is a hypersurface of degree d.

Therefore, X is cut out by a polynomial

$$\sum_{i_0+...+i_d} b_{i_0,...,i_d} x_0^{i_0} ... x_d^{i_d}$$

on $B \times \mathbb{P}^r$, where the $b_{i_0,...,i_d}$ are in $\mathrm{Spec}(B)$. This is the pullback of the proposed universal family. So the Hilbert scheme $\mathcal{H}_{p(t),r}$ is just the projective space $\mathbb{P}^m$.

A similar example, not carried out here, is the Hilbert scheme parametrizing k-planes in $\mathbb{P}^r$, which is isomorphic to the corresponding Grassmannian scheme. Again, the Hilbert scheme is the same as the parameter space that one may expect. As a special case, the Hilbert scheme of lines in $\mathbb{P}^r$ is a projective space of dimension $2r-2$.

The next examples are curves in $\mathbb{P}^r$. In this case, the Hilbert polynomial is determined by the degree d and the genus g of the curve. The notation $\mathcal{H}_{d,g,r}$ is clearer than $\mathcal{H}_{dn+(1-g),r}$ and appears more commonly in the literature.

Plane Conics: The Hilbert scheme $\mathcal{H}_{2,0,3}$ of conics in $\mathbb{P}^3$ is a simple example. Every curve of degree 2 is a complete intersection of a plane and a quadric surface. Because a conic is contained in a unique plane, there is a map of sets

$$\mathcal{H}_{2,0,3} \to (\mathbb{P}^3)^*.$$

The fiber over a plane H is the space of conics in H, which is parametrized by the projective space $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2)) \cong \mathbb{P}^5$ of degree 2 polynomials on H. As the fibers over $(\mathbb{P}^3)^*$ are isomorphic to $\mathbb{P}^5$, the reduced scheme $(\mathcal{H}_{2,0,3})_{\mathrm{red}}$ is an eight-dimensional projective space. To show that $\mathcal{H}_{2,0,3} \cong \mathbb{P}^8$, it is sufficient to check that the tangent space at every point is eight-dimensional. This straightforward calculation will be carried out in Section 2.4.

Note that while a general point of the Hilbert scheme parametrizes a smooth conic, there are also points parametrizing the union of two distinct lines meeting at a point or a double line of genus 0. Because every singular conic is a limit of smooth conics, the points parametrizing singular conics are also limits of points corresponding to smooth conics. This is because any point is a limit of a sequence in the Hilbert scheme if and only if the curve parametrized by that point is a flat limit of the corresponding sequence of curves.

Twisted Cubics: The case of curves of degree 3 and genus 0 in $\mathbb{P}^3$ is analogous, but more interesting. The first issue is that not every rational cubic is irreducible. Because a plane curve of degree 3 has genus 1, the union of a plane cubic C and a distinct point p has genus 0. The Hilbert scheme does contain irreducible curves as well, twisted cubics, and these are non-planar.

Note that a twisted cubic is not contained in a plane but is irreducible, so a twisted cubic cannot be a limit of curves $C \cup \{p\}$ and a curve $C \cup \{p\}$ cannot be a limit of twisted cubics. This implies that the two types of curves give distinct components of the Hilbert scheme.

Describing plane curves of degree 3 is similar to the previous example, plane conics. A computation shows that plane cubics are parametrized by a $\mathbb{P}^{12}$. The component of $\mathcal{H}_{3,0,3}$ turns out to be the blow-up of the product of this $\mathbb{P}^{12}$ with $\mathbb{P}^3$ along the subset $\{(C,p) : p \in C\}$. Each curve in this subset has an embedded point, which requires an additional variable specifying a tangent direction.

There are a number of approaches to describing the global structure of the component containing twisted cubics. The simplest is to observe that every rational curve is the image of a map from $\mathbb{P}^1$ to $\mathbb{P}^3$. This map can be given in projective coordinates as

$$(x_0, x_1) \mapsto (f_0(x_0, x_1), f_1(x_0, x_1), f_2(x_0, x_1), f_3(x_0, x_1)),$$

where each f_i is a degree 3 polynomial. The choices of polynomials are given by 16 parameters. These choices determine a twisted cubic up to scalar multiplication and automorphisms of $\mathbb{P}^1$. The automorphisms of $\mathbb{P}^1$ are the 3-dimensional group $PGL(2)$, so the component has dimension 12.

Another approach relies on the simple geometric fact that there is a unique twisted cubic passing through 6 points in general position. Because the space of six-tuples of points in general position is 18-dimensional, so is the incidence correspondence

$$\{(C, p_1, ..., p_6) : p_i \in C\}.$$

This set projects to the space of twisted cubics with 6-dimensional fibers. A third description gives a twisted cubic as a determinantal scheme.

The component containing smooth twisted cubics curves contains many more types of curves, including curves with embedded points and curves with a component of multiplicity greater than one. Harris describes each possible limit of twisted cubics and states which curves are limits of each family [3].

2.3 Construction of the Hilbert Scheme

The basic idea of the construction is that the functor $\mathbf{Hilb}_{p(n),r}$ can be represented by as a subscheme of a sufficiently large Grassmannian scheme. The Grassmannian scheme parametrizes k-planes contained in a projective space $\mathbb{P}^r$. For discussions of the Grassmannian scheme as a projective scheme and as a representable functor, see for example [2].

The Hilbert scheme parametrizes schemes in $\mathbb{P}^r$ with the Hilbert polynomial $p(n)$. In sufficiently high degree n, each scheme X is determined by the twists $\mathcal{I}_X(n)$ of its ideal sheaf. Also assume that n is large enough so that the global sections $h^0(\mathcal{I}_X(n))$ are a subspace of the homogeneous polynomials $h^0(\mathcal{O}(n))$ of degree n. This subspace determines a point in a Grassmannian. The complicated step is showing that n can be chosen independent of the scheme X. The construction, given with several unproven lemmas here, is due to Grothendieck [7].

Fix $X \subset \mathbb{P}^r$ with Hilbert polynomial $p(n)$. For sufficiently large n, the dimension of the ideal of functions vanishing on X is

$$h^0(\mathcal{I}_X(n)) = h^0(\mathcal{O}(n)) - p(n) = \binom{r+n}{n} - p(n).$$

The ideal sheaf uniquely fixes the subscheme X.

Lemma 2.3.1. Given a projective space $\mathbb{P}^r$ and a Hilbert polynomial $p(n)$, there is an integer d_0 depending only on r and $p(n)$ such that whenever $d \geq d_0$ and $X \subset \mathbb{P}^r$ has Hilbert polynomial $p(n)$, the ideal of X is generated by polynomials of degree at most d and

$$h^0(\mathcal{I}_X(d)) = h^0(\mathcal{O}(d)) - p(d).$$

Therefore, each scheme X is associated to the point in the Grassmannian

$$G = G(h^0(\mathcal{O}(d_0)) - p(d_0), h^0(\mathcal{O}(d_0)))$$

parametrizing $h^0(\mathcal{I}_X(d_0))$. A second lemma specifies the subscheme of such points.

Lemma 2.3.2. Suppose $d \geq d_0$, where d_0 is as in Lemma 2.3.1. A subspace I of codimension $p(d)$ in $h^0(\mathcal{O}(d))$ generates the ideal of a subscheme $X \subset \mathbb{P}^r$ with Hilbert polynomial $p(d)$ if and only if

$$\dim H^0(\mathcal{O}(d+1))/(I \otimes \mathcal{O}(1)) \geq p(d+1),$$

where $H^0(\mathcal{O}(d+1))/(I \otimes \mathcal{O}(1))$ is the quotient of the degree $(d+1)$-polynomials by the image of I under multiplication by linear polynomials.

Given this criterion, $\mathcal{H}_{p(n),r}$ can be expressed as a determinantal scheme in G. Let $\mathcal{U}$ be the universal sub-bundle in the trivial vector bundle $\mathcal{O}(d) \otimes \mathcal{O}_G$. This is the bundle with sections over each point in G given by the subspace parametrized by that point. Inclusion and then multiplication give maps

$$\mathcal{O}(1) \otimes \mathcal{U} \to \mathcal{O}(1) \otimes \mathcal{O}(d) \otimes \mathcal{O}_G \to \mathcal{O}(d+1) \otimes \mathcal{O}_G.$$

The composition has co-rank at least $p(d+1)$ exactly at the points of G parametrizing subspaces I satisfying

$$\dim H^0(\mathcal{O}(d+1))/(I \otimes \mathcal{O}(1)) \geq p(d+1).$$

This determines $\mathcal{H}_{p(n),r}$ as a determinantal subscheme $\mathcal{H}$ of G.

Because this determinantal scheme parametrizes schemes with Hilbert polynomial $p(n)$, it should at least be plausible that it is the correct scheme and satisfies the correct universal property. The remainder of the proof is the following lemma:

Lemma 2.3.3. The determinantal subscheme $\mathcal{H}$ given above gives rise to a natural family $\mathscr{H} \subset G \times \mathbb{P}^r$, and the restriction to $\mathcal{H} \times \mathbb{P}^r$ is the universal family described in Section 2.1.

2.4 The Tangent Space to the Hilbert Scheme

The characterization of the Hilbert scheme as a representable functor gives a particularly simple description of the tangent space to the Hilbert scheme at a point. This is a key tool in describing the geometry of the Hilbert scheme. Most simply, comparing the dimension of a component to the dimension of the tangent space often shows that a Hilbert scheme is smooth at a given point.

It is sometimes unclear whether a family of schemes is dense in a component of a Hilbert scheme. For example, families of linearly equivalent curves on a quadric surface can be dense in a component or can have positive codimension, depending on the divisor class. Here, comparing the dimension of the family to the dimension of the tangent space to the whole component of the Hilbert scheme can show that the family is dense.

These computations can also demonstrate degenerate behavior in the geometry of the Hilbert scheme. David Mumford showed that there is a component of the Hilbert scheme of curves of degree 14 and genus 24 in $\mathbb{P}^3$ with dimension 56 but with a 57-dimensional tangent space at every point. This implies that the Hilbert scheme is nowhere smooth, so it is not reduced. Moreover, the generic curve in this component is smooth [8]!

Suppose $p \in \mathcal{H}_{p(n),r}$ parametrizes a scheme $X \subset \mathbb{P}^r$. Then the central result is the following theorem:

Theorem 2.4.1. *The tangent space to the Hilbert scheme $\mathcal{H}_{p(n),r}$ at p is isomorphic to the global sections $H^0(N_{X/\mathbb{P}^r})$ of the normal sheaf of X in $\mathbb{P}^r$.*

The normal sheaf is given by the simple algebraic definition

$$N_{X/Y} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_X),$$

where $\mathcal{I}$ is the ideal of X in Y.

In the case where X and Y are smooth, the normal sheaf is a vector bundle and fits into an exact sequence

$$0 \to T_Y \to T_X|_Y \to N_{X/Y} \to 0$$

with the tangent bundles of X and Y. The geometric interpretation is the bundle of normal vectors to a variety.

The idea of the proof is to show that $T_{\mathcal{H}_{p(n),r},p}$ and $H^0(N_{X/\mathbb{P}^r})$ are both isomorphic to the space of first-order deformations of X in $\mathbb{P}^r$.

Definition 2.4.2. If $X \subset Y$ is a closed subscheme, then a first-order deformation of X in Y is a flat family $\mathscr{X} \subset Y \times \mathrm{Spec}\ \mathbb{C}[\epsilon]/\epsilon^2$ such that the fiber over Spec $\mathbb{C}$ is X.

The theory of Hilbert schemes is closely tied to deformation theory in a number of ways. For an introduction to deformation theory and more discussion of some of these connections, see for example [5].

The proof of Theorem 2.4.1 will follow easily from two lemmas.

Lemma 2.4.3. For any scheme X, then the tangent space T_pX at a point $p \in X$ is the space of maps Spec $\mathbb{C}[\epsilon]/\epsilon^2 \to X$ sending 0 to p.

Proof. Fix a map Spec $\mathbb{C}[\epsilon]/\epsilon^2 \to X$ sending 0 to p. Such a map induces a homomorphism $\mathcal{O}_{X,p} \to \mathbb{C}[\epsilon]/\epsilon^2$. The homomorphism is local because every function vanishing at p must map to a function vanishing at ϵ, so it restricts to a map from the maximal ideal $\mathfrak{m}_{X,p}$ of the local ring to the maximal ideal (ϵ). Because $\epsilon^2 = 0$, any map from $\mathfrak{m}_{X,p}$ to (ϵ) gives rise to a map $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2 \to (\epsilon) \cong \mathbb{C}$. Because the tangent space is $\mathrm{Hom}(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2, \mathbb{C})$, this is an element of the tangent space.

To go in the opposite direction, fix an element $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2 \to \mathbb{C}$ of the tangent space. The projection onto the residue field gives a map $\mathcal{O}_{X,p} \to \mathbb{C}$. The projection gives a splitting of $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^2$ as a direct sum $\mathbb{C} \oplus \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$. Therefore, the identity map and the tangent space element give a map $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^2 \to \mathbb{C}[\epsilon]/\epsilon^2$. Composing with the natural map from $\mathcal{O}_X$ gives a map

$$\mathcal{O}_X \to \mathbb{C}[\epsilon]/\epsilon^2$$

such that the corresponding map Spec $\mathbb{C}[\epsilon]/\epsilon^2 \to X$ sends 0 to p.

It is easy to see that these constructions are inverses, so the two spaces are identified as sets. Moreover, the space of maps Spec $\mathbb{C}[\epsilon]/\epsilon^2 \to X$ has a natural vector space structure that agrees with the vector space structure on T_pX.

The subtle point is how to define addition. Any two distinct maps Spec $\mathbb{C}[\epsilon]/\epsilon^2 \to X$ induce a map Spec $\mathbb{C}[\epsilon, \epsilon']/(\epsilon, \epsilon')^2 \to X$ from their fiber product to X.

There is a homomorphism $\mathbb{C}[\epsilon, \epsilon']/(\epsilon, \epsilon')^2 \to \mathbb{C}[\epsilon]/\epsilon^2$ sending ϵ and ϵ' to ϵ. This homomorphism gives a map Spec $\mathbb{C}[\epsilon]/\epsilon^2 \to$ Spec $\mathbb{C}[\epsilon, \epsilon']/(\epsilon, \epsilon')^2$, and composing with the map to X gives the sum. It is straightforward to define scalar multiplication and check that both operations agree with the vector space structure on the tangent space T_pX. □

Now, take the scheme to be $\mathcal{H}_{p(n),r}$ in this lemma and suppose $p \in \mathcal{H}_{p(n),r}$ parametrizes $X \subset \mathbb{P}^r$. By the universal property of the Hilbert scheme, there is a bijective correspondence between maps Spec $\mathbb{C}[\epsilon]/\epsilon^2 \to \mathcal{H}_{p(n),r}$ sending 0 to p and flat families $\mathscr{H} \subset \mathbb{P}^r \times$ Spec $\mathbb{C}[\epsilon]/\epsilon^2$ such that the fiber over Spec $\mathbb{C}$ is X. But this is the space of first-order deformations of X in $\mathbb{P}^r$, so the tangent space $T_p\mathcal{H}_{p(n),r}$ is the space of first-order deformations of X in $\mathbb{P}^r$. By identifying the first-order deformations with the normal bundle $N_{X/\mathbb{P}^r}$, the second lemma completes the proof.

Lemma 2.4.4. If $X \subset \mathbb{P}^r$ is a space of first-order deformations of X in $\mathbb{P}^r$ is isomorphic to the sections of the normal sheaf $h^0(N_{X/\mathbb{P}^r})$.

Proof. Suppose that Y is affine and $X \subset Y$ is a closed subscheme. Because Y is affine, the closed subscheme X is cut out by an ideal $I \subset \mathcal{O}_Y$. The normal sheaf of X in Y is given by

$$N_{X/Y} = \mathrm{Hom}_{\mathcal{O}_Y}(I, \mathcal{O}_Y/I).$$

The ring of regular functions on $Y \times \mathrm{Spec}\ \mathbb{C}[\epsilon]/\epsilon^2$ is equal to the tensor product $\mathcal{O}_Y \otimes \mathbb{C}[\epsilon]/\epsilon^2$. Every element of this ring has the form $f + \epsilon g$, where f and g are elements of $\mathcal{O}_Y$.

Fix an arbitrary subscheme $\mathscr{X} \subset Y \times \mathrm{Spec}\ \mathbb{C}[\epsilon]/\epsilon^2$, not necessarily a flat family over Spec $\mathbb{C}[\epsilon]/\epsilon^2$. The ideal I' of $\mathscr{X}$ in $\mathcal{O}_Y$ is generated by

$$(f_1 + \epsilon g_1, ..., f_n + \epsilon g_n),$$

where the elements $f_1, ..., f_n$ generate the ideal I. The key fact to show is that there is an $\mathcal{O}_Y$-module homomorphism $\phi \in \mathrm{Hom}(I, \mathcal{O}_Y/I)$ sending f_i to g_i if and only if $\mathscr{X}$ is a flat family over Spec $\mathbb{C}[\epsilon]/\epsilon^2$.

Basic commutative algebra states that a module M over a ring R is flat if and only if multiplication $I \otimes M \to M$ is injective for every ideal $I \subset R$ [9]. Here, the $\mathbb{C}[\epsilon]/\epsilon^2$ module M is flat if and only if tensoring with M preserves exactness of the sequence

$$0 \to (\epsilon) \to \mathbb{C}[\epsilon]/\epsilon^2 \to \mathbb{C} \to 0.$$

So the subscheme $\mathscr{X}$ is flat if and only if $(\epsilon) \otimes \mathcal{O}_{\mathscr{X}} \to (\epsilon)$ is injective, or equivalently $f \in I$ whenever ϵf is in I'. There is an expression

$$\epsilon f = \sum (a_i + \epsilon b_i)(f_i + \epsilon g_i) = \sum a_i f_i + \epsilon(\sum a_i g_i + b_i f_i),$$

and $\sum a_i f_i = 0$ because that term is not divisible by ϵ while all others are.

Suppose there exists an $\mathcal{O}_Y$-module homomorphism ϕ sending f_i to g_i. Then

$$\sum a_i g_i = \sum a_i \phi(f_i) = \phi(\sum a_i f_i) = 0.$$

Therefore, $f = \sum b_i f_i \in I$ whenever $\epsilon f \in I'$, which implies that $\mathscr{X}$ is a flat family by the previous condition.

Suppose that $\mathscr{X}$ is a flat family over Spec $\mathbb{C}[\epsilon]/\epsilon^2$ cut out by

$$\{f_1 + \epsilon g_1, ..., f_n + \epsilon g_n\}.$$

By the previous criterion for flatness, $f \in I$ whenever $\epsilon f \in I'$. Define a homomorphism $I \to \mathcal{O}_Y/I$ by sending f_i to g_i. To check this is well-defined, suppose that $\sum a_i f_i = 0$ for some $a_i \in \mathcal{O}_Y$. Then because

$$\epsilon \sum a_i g_i = \sum a_i (f_i + \epsilon g_i) \in I',$$

the sum $\sum a_i g_i \in I$. The homomorphism is well-defined as a map to $\mathcal{O}_Y/I$.

The final step is to pass from the affine case to a general scheme Y. Suppose Y is covered by affine open subschemes U. If $\mathscr{X}$ is flat, there is a homomorphism $\phi|_U$ mapping $I(U)$ to $\mathcal{O}_Y(U)/I(U)$ for each U. These agree on the intersections $U \cap U'$, so the flat family gives an element of the normal sheaf $N_{X/Y}$. If $\phi \in N_{X/Y}$, then the ideal

$$\{f + \epsilon\phi(f) : f \in I\}$$

cuts out a flat family in $Y \times \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$. □

The proof is complete. As an example, consider a complete intersection $C = S \cap T$ of smooth surfaces of degrees s and t. The normal sheaf of C in $\mathbb{P}^3$ is the direct sum

$$N_{C/\mathbb{P}^3} = \mathcal{O}(s) \oplus \mathcal{O}(t).$$

If f cuts out S and g cuts out T, then a homomorphism $\mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^3}/\mathcal{I}_C$ is determined by the image of f and g. The image of f can be any element of $\mathcal{O}(s)$ and the image of t can be any element of $\mathcal{O}(t)$, so the normal sheaf is their direct sum. For more discussion of how to compute normal bundles, see [10].

The computation that the space of conics in $\mathbb{P}^3$ is smooth can be completed now. The tangent space at a point $p \in \mathcal{H}_{2,0,3}$ parametrizing a conic C is

$$T_p(\mathcal{H}_{2,0,3}) = H^0(N_{C/\mathbb{P}^3}) = H^0(\mathcal{O}(1) \oplus \mathcal{O}(2)).$$

By Riemann-Roch, $\mathcal{O}(1)$ is 3-dimensional and $\mathcal{O}(2)$ is 5-dimensional on C. The tangent space is eight-dimensional, as is the Hilbert scheme, so the Hilbert scheme is smooth.

Chapter 3

Techniques and Methods Using Surfaces

While the examples in Section 2.2 were relatively simple, computing the number of components of the restricted Hilbert scheme and their dimensions is a very difficult problem in general. First, the best existing bounds on the degrees of curves where there is only one Hilbert scheme component parametrizing smooth, non-degenerate curves are likely not close to being sharp. Moreover, even when the number of components is clear, their dimensions need not be. While the expected dimension does give a general lower bound, the actual dimension is much larger for many classes of curves.

This chapter outlines basic techniques for these computations that focus on the surfaces containing a curve. Before addressing these techniques, a definition is helpful. The Hilbert scheme can parametrize many degenerate curves, such as unions of curves and distinct points or reducible curves. While any component of the Hilbert scheme will have degenerate curves on the boundary, it makes sense to require that a generic point be reasonably well-behaved.

Definition 3.0.5. The restricted Hilbert scheme $\mathcal{I}'_{d,g,r}$ of curves of degree d and genus g in $\mathbb{P}^r$ is the union of the components of the Hilbert scheme $\mathcal{H}_{d,g,r}$ such that a generic point parametrizes a smooth, non-degenerate curve.

Note that the terminology is not standard, but the notation $\mathcal{I}'_{d,g,r}$ is. Restricted Hilbert schemes will be the main object of study in the subsequent discussions.

3.1 Complete Intersections

Complete intersections of surfaces in $\mathbb{P}^3$ are among the simplest cases. Let S and T in $\mathbb{P}^3$ be surfaces of degree s and t, respectively. The intersection of the two surfaces is a curve of degree st by Bézout's Theorem.

The genus of a complete intersection is straightforward to compute from the adjunction formula. By adjunction, the canonical divisor on C is given by

$$\omega_C = \omega_S(t) \otimes \mathcal{O}_C.$$

The canonical divisor ω_S on the surface S is $\mathcal{O}_{\mathbb{P}^3}(s-4)$, so

$$\omega_C = \mathcal{O}_C(s+t-4).$$

Taking degrees,

$$2g - 2 = st(s+t-4),$$

so $g = \frac{1}{2}st(s+t-4) + 1$.

Moreover, the family of complete intersections of surfaces of degree S and T is a dense subset of an irreducible component of the restricted Hilbert scheme. If $s = t$, then the family of complete intersection curves is the Grassmannian scheme $G(2, \binom{s+3}{3})$ of pencils in the projective space parametrizing degree s surfaces. The Grassmannian is irreducible of dimension $2(\binom{s+3}{3} - 2)$.

If $s < t$, then fix a surface S cut out by a polynomial f. Each surface T determines a complete intersection curve, and T determines the same curve as T' if and only if the two surfaces are cut out by polynomials differing by a multiple of f. The set of surfaces of degree t, determined up to adding multiples of f, is a projective space of dimension $\binom{t+3}{3} - \binom{t-s+3}{3} - 1$. Therefore, the family of complete intersection curves is a $\mathbb{P}^{\binom{t+3}{3} - \binom{t-s+3}{3} - 1}$ bundle over $\mathbb{P}^{\binom{s+3}{3}-1}$. Such a bundle must be an irreducible space of dimension

$$\binom{s+3}{3} + \binom{t+3}{3} - \binom{t-s+3}{3} - 2.$$

The family of complete intersection curves is dense in a component of the restricted Hilbert scheme. One approach is to compute the dimension of the normal bundle $N_{C/\mathbb{P}^3}$ and show that this dimension is equal to the dimension of the family of complete intersection curves. Recall from Section 2.4 that

$$N_{C/\mathbb{P}^3} = \mathcal{O}(s) \oplus \mathcal{O}(t).$$

Another, taken here, is to argue directly that a complete intersection of surfaces S and T cannot be the limit of curves which are not complete intersections.

It is sufficient to show that a smooth complete intersection curve C cannot be a limit of curves C_t which are not complete intersections. Let $\mathcal{I}_t$ be the ideals of C_t and $\mathcal{I} = (f, g)$ be the ideal of $C = C_0$. Then there must exist elements $f_t \in \mathcal{I}_t$ of the same degree as f such that $f_t \to f$ as $t \to 0$. Similarly, there must exist elements $g_t \in \mathcal{I}_t$ of the same degree as g such that $g_t \to g$ as $t \to 0$ and g_t has no common factors with f_t for each t. But C_t is contained in surfaces of degree $\deg f$ and $\deg g$. Because $\deg C_t = \deg f \deg g$, each C_t must be the complete intersection of these surfaces, as desired.

Though irreducible, the family of complete intersections of degree s and t surfaces need not be the only component. In Section 5.2, there is an example of a restricted Hilbert scheme containing a component of complete intersection curves as well as another component.

Finally, the above statements generalize easily to complete intersections of $r - 1$ hypersurfaces in $\mathbb{P}^r$. The expressions of the dimension grow more complicated, of course, but remain tractable.

3.2 Curves on Quadric Surfaces

If a curve lies on a surface of very low degree, then the geometry of that surface can give insights about the restricted Hilbert scheme parametrizing that curve. This is most effective for quadric and cubic surfaces, which have well-known Picard groups.

Suppose Q is a smooth degree two surface in $\mathbb{P}^3$. The Picard group on Q is the free Abelian group $\mathbb{Z}^2$, and the two generators are lines of distinct rulings. One way to see this is to consider the Segre embedding $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ sending (x_0, x_1, y_0, y_1) to $(x_0y_0, x_0y_1, x_1y_0, x_1y_1)$. The image is the quadric surface cut out by $xw - yz$. Because all quadric surfaces are projectively equivalent, it is sufficient to compute the Picard group of $Q = xw - yz$.

Every curve on Q is cut out by a bihomogeneous polynomial on $\mathbb{P}^1 \times \mathbb{P}^1$. Call a curve of type (a, b) if the polynomial has degree a in the x_i and degree b in the y_i. The polynomials of type $(1, 0)$ and type $(0, 1)$ cut out lines which intersect exactly once. Any two curves of type (a, b) are linearly equivalent, so these lines generate the Picard group. For example, a curve of type (a, a) is the complete intersection of Q with a hyperplane of degree a.

The degree of a curve of type (a, b) is $a + b$. The genus of any curve on a smooth quadric follows from a general formula for the genus of a smooth curve C of degree d on a smooth surface S of degree s. The derivation is clearest with the language of intersection pairings on S, which is introduced in sources such as [11]. By the adjunction formula,

$$\omega_C = \mathcal{O}_C \otimes \omega_S(C) = \mathcal{O}_C \otimes \mathcal{O}_{\mathbb{P}^3}(s-4)(C).$$

Taking degrees and solving,

$$g = \frac{C^2 + (s-4)d}{2} + 1.$$

In the case of interest, $d = a + b$, $s = 2$, and $C^2 = 2ab$ because each line has self-intersection zero and the two lines of distinct rulings intersect once. So the genus is $g = ab - a - b + 1 = (a-1)(b-1)$.

Assume that $a+b \geq 5$ so that each curve of type (a, b) lies on just one quadric. The family of curves of type (a, b) on a fixed quadric has dimension $(a+1)(b+1)-1$, because specifying a curve is equivalent to specifying a polynomial of bidegree (a, b) up to scalar multiplication and the space of bidegree (a, b) polynomials is generated by $(a+1)(b+1)$ linearly independent monomials. The family of quadrics in $\mathbb{P}^3$ is 9-dimensional, so the family of all curves of degree $a + b$ and genus $(a-1)(b-1)$ lying on smooth quadrics has dimension $(a+1)(b+1) + 8$.

Next, the family of curves of a fixed degree and genus lying on quadrics is irreducible. It is clear that the components of curves of type (a, b) and type (b, a) are both irreducible, and in fact both are contained in the same component. This is because as the surfaces Q_t varies in a flat family, the two rulings are exchanged.

Because all quadric surfaces are projectively equivalent, it is sufficient to check this theorem for the surface $Q = xy - zw$. The ruling can be given by the two lines $L = (x, z)$ and $L' = (x, w)$, because the intersection $L \cap L'$ is the point $(0, 1, 0, 0)$. Rotating the surface $\pi/2$ radians around the line (x, y) exchanges the two lines L and L'. So every curve of type (b, a) on Q is a flat limit of curves of type (a, b) on Q, and conversely every curve of type (a, b) is a flat limit of curves of type (b, a).

Moreover, its closure contains any such curve lying on a quadric cone. The proof relies on the geometry of the Hirzebruch surface $\mathbb{F}_2$. In general, $\mathbb{F}_n$ is the projectivization $\mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(-n))$ of the sheaf $\mathcal{O}(0) \oplus \mathcal{O}(-n)$ on $\mathbb{P}^1$.

Blowing up a quadric cone $Q \subset \mathbb{P}^3$ at the cone point gives the Hirzebruch surface $\mathbb{F}_2$. The short exact sequence of the blow-up gives

$$0 \to \mathbb{Z} \cdot E \to \mathrm{Pic}(\mathbb{F}_2) \to \mathrm{Pic}(Q) \to 0,$$

where E is the exceptional divisor. The Picard group of $\mathbb{F}_2$ is isomorphic to $\mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \epsilon$ where f is a fiber and ϵ is a section of the surface. It follows that $\mathrm{Pic}(Q) \cong \mathbb{Z}$ is generated by a line L. In particular, the hyperplane section is $H = 2L$.

Suppose $C \subset Q$ is a curve lying on the cone. If the curve has even degree $\deg C = 2k$, then C is linearly equivalent to the divisor kH. This means that $C = Q \cap S$ for a surface S of degree k. Then choosing any flat family Q_t of quadric surfaces such that Q_t is smooth for $t \neq 0$ and $Q_0 = Q$, the curve C is the flat limit of $Q_t \cap S$.

If the curve has odd degree $\deg C = 2k+1$, then $C + L$ is linearly equivalent to the divisor $(k+1)H$. So C is residual to a line L' in the complete intersection of Q and a surface S of degree $k+1$. Choose a flat family of conics Q_t such that Q_t is smooth for $t \neq 0$, $Q_0 = Q$ and $L' \subset Q_t$. For each t, the line L' is residual to a curve C_t in the intersection $Q_t \cap S$. Then $C_t \to C$, so C is the limit of curves of smooth quadrics.

Finally, if $a \geq 3$ and $b \geq 3$, then the family of curves of type (a, b) on quadrics is dense in an irreducible component of the Hilbert scheme. As a counterexample when $b < 3$, a rational quintic can have type $(4, 1)$ on a quadric. But this family is 18-dimensional, while the restricted Hilbert scheme parametrizing all rational quintics is 20-dimensional.

When a and b are at least three, computing the tangent space to the Hilbert scheme at a point p parametrizing a smooth curve C of type (a, b) shows these curves are dense in a component. By Section 2.4, this tangent space is isomorphic to the sections of the normal bundle $N_{C/\mathbb{P}^3}$.

The key tool for computing this bundle is the short exact sequence of the curve C on the surface Q:

$$0 \to \mathcal{O}_Q \to \mathcal{O}_Q(C) \to \mathcal{O}_C(C) \to 0.$$

Before beginning, it is helpful to find the ranks of the cohomology groups of $\mathcal{O}_Q(k, l)$. The space of sections of $\mathcal{O}_Q(k, l)$ is the vector space of bihomogeneous polynomials of degree a in the first variable and degree b in the second variable, which has sections of dimension

$$h^0(\mathcal{O}_Q(k, l)) = (a+1)(b+1) \text{ for } a, b \geq 0 \text{ and } h^0(\mathcal{O}_Q(k, l)) = 0 \text{ for } a < 0 \text{ or } b < 0.$$

By the Kunneth formula,

$$h^1(\mathcal{O}_Q(k, l)) = 0$$

whenever $k, l \geq -1$. By Serre duality,

$$h^1(\mathcal{O}_Q(k, l)) = 0$$

whenever $k, l \leq -1$ as well.

The short exact sequence of the curve C on the surface Q gives rise to a short exact sequence of normal bundles

$$0 \to N_{C/Q} \to N_{C/\mathbb{P}^3} \to N_{Q/\mathbb{P}^3}|_C \to 0.$$

The short exact sequence gives rise to a long exact sequence in cohomology:

$$0 \to H^0(N_{C/Q}) \to H^0(N_{C/\mathbb{P}^3}) \to H^0(N_{Q/\mathbb{P}^3}|_C) \to H^1(N_{C/Q}) \to \dots$$

Determining $h^0(N_{C/\mathbb{P}^3})$ is a matter of understanding the surrounding terms in this sequence.

The normal bundle of the surface $N_{Q/\mathbb{P}^3}$ is $\mathcal{O}(2)$, so the third term is $H^0(\mathcal{O}_C(2))$. Twisting the short exact sequence of C on Q gives a long exact sequence

$$0 \to H^0(\mathcal{O}_Q(2-C)) \to H^0(\mathcal{O}_Q(2)) \to H^0(\mathcal{O}_C(2)) \to H^1(\mathcal{O}_Q(2-C)) \to \dots$$

The dimensions of these groups were computed above. Since a and b are both at least 3, $\mathcal{O}_Q(2-C) = \mathcal{O}_Q(k,l)$ with k and l negative. The first cohomology group vanishes because $h^1(\mathcal{O}_Q(k,l)) = 0$ for k and l both negative. Because $h^0(\mathcal{O}_Q(2)) = 9$ and $h^0(\mathcal{O}_Q(2-C)) = 0$, the sections of $\mathcal{O}_C(2)$ have dimension

$$h^0(\mathcal{O}_C(2)) = 9.$$

The normal bundle $N_{C/Q}$ of C in Q is $\mathcal{O}_C(C) = \mathcal{O}_C(a,b)$. The short exact sequence

$$0 \to \mathcal{O}_Q \to \mathcal{O}_Q(C) \to \mathcal{O}_C(C) \to 0,$$

is exact on global sections because $h^1(\mathcal{O}_Q) = 0$. Substituting for the dimensions of the global sections of each term,

$$h^0(\mathcal{O}_C(a,b)) = h^0(\mathcal{O}_Q(a,b)) - 1 = (a+1)(b+1) - 1.$$

Finally, $h^1(N_{C/Q}) = h^1(\mathcal{O}_C(C))$ is zero. This follows from the same sequence, because $h^1(\mathcal{O}_Q(C)) = 0$ and $h^2(\mathcal{O}_Q) = 0$.

Adding dimensions, the conclusion is that

$$\dim T_p(\mathcal{H}_{d,g,3}) = h^0(N_{C/\mathbb{P}^3}) = (a+1)(b+1) + 8.$$

The dimension of the space of quadric curves is also $(a+1)(b+1)+8$, because the family of quadric surfaces in $\mathbb{P}^3$ is 9-dimensional and the family of curves of degree d and genus g on each surface has dimension $(a+1)(b+1)-1$. This family must be dense in an irreducible component of the Hilbert scheme.

The description of a curve lying on a smooth cubic surface is similar, albeit more complicated. A smooth cubic surface is isomorphic to the plane blown up at six points, and the Picard group is generated by a line in the plane away from the blow-up points and the six exceptional divisors [11]. The degree and genus of a curve can be computed from its divisor class, and divisor classes satisfying simple conditions contain smooth curves [1]. For an important application of these methods, see Mumford [8].

3.3 Liaison of Curves on Surfaces

When information about the surfaces containing a curve about $\mathbb{P}^3$ is known, the method of liaison is a powerful tool. The basic idea is that if the complete intersection of two surfaces is the union of two curves, then the properties of the two curves can be related. If the genus and degree of one curve is known, a simple formula gives the degree and genus of the residual curve. Moreover, the method can relate restricted Hilbert schemes as well as certain cohomology groups.

The basic setup is as follows. Let C be a curve of degree d and genus g, and suppose C is contained in surfaces S of degree s and T of degree t. If C is not a complete intersection, then there is a curve D such that $S \cap T = C \cup D$.

Definition 3.3.1. A curve C is (geometrically) linked to a curve D if the two curves have no common components, and $C \cup D = S \cap T$ is a scheme-theoretic complete intersection.

If the linked curve has degree d' and genus g', then the result is that

$$2(g - g') = (s + t - 4)(d - d').$$

This formula is easiest to prove for C and D smooth curves and S and T smooth surfaces, but these hypotheses can be relaxed.

By the formula for the genus of a curve on a surface,

$$g = \frac{C^2 + (s-4)d}{2} + 1.$$

Solving for the self-intersection of C gives

$$C^2 = 2g - 2 - (s-4)d.$$

Because $C + D = tH$, this gives

$$C \cdot D = C \cdot (tH - C) = td - (2g-2) + (s-4)d = (s+t-4)d - (2g-2).$$

Analogously,

$$D^2 = 2g' - 2 - (t-4)d',$$

and this expression of the self-intersection implies

$$C \cdot D = (s+t-4)d' - (2g'-2).$$

Setting the two expressions for $C \cdot D$ equal, the conclusion is that

$$(s+t-4)(d-d') = 2(g-g').$$

A useful technique for describing the Hilbert scheme component is to link the curves parametrized by its elements to curves in another Hilbert scheme component that is better understood. In general, linkage is most useful in cases where $\mathcal{H}_{d,g,3}$ can be related to a Hilbert scheme parametrizing curves of lower degree and genus.

The technique of linkage will be used extensively in subsequent proofs, but a very simple example is given here. Consider the restricted Hilbert scheme of curves of degree 4 and genus 1 in $\mathbb{P}^3$. By Riemann-Roch, such a curve C lies on a $\mathbb{P}^1$ of quadric surfaces and at least a $\mathbb{P}^7$ of cubic surfaces. Because each quadric is contained in only a $\mathbb{P}^3$ of reducible cubics, the curve must lie on irreducible cubics. These curves are actually complete intersection of two quadrics, but linkage gives another description.

If D is the residual curve in the intersection of a quadric Q and an irreducible cubic S, then the formulas above give

$$d + d' = 5 \text{ and } (3 + 2 - 4)(d - d') = 2(g - g').$$

Solving for d' and g' shows that D is a plane conic. Riemann-Roch predicts that a plane conic will lie on a $\mathbb{P}^4$ of conics and a $\mathbb{P}^{12}$ of cubic surfaces. The prediction is correct, and any plane conic does lie on the predicted spaces of conics and cubics. Because all plane conics are projectively equivalent, it is sufficient to check for a single curve, which is easy. Alternately, the general fact that any complete intersection lies on the expected number of surfaces in each degree is Corollary 3.3.3 below.

The next step is to consider the incidence correspondence of four-tuples

$$(C, D, Q, S)$$

such that $C \cup D = Q \cap S$, where C is a curve of degree 4 and genus 1, D is a plane conic, Q is a quadric surface and S is an irreducible cubic surface. The incidence correspondence

$$\Phi = \{(C, D, Q, S) : C \cup D = Q \cap S\}$$

projects onto the restricted Hilbert scheme $\mathcal{I}'_{2,0,3}$ by projection onto D. A projective bundle over an irreducible scheme is irreducible. Because

$$\Phi' = \{(D, Q) : D \subset Q\}$$

is a projective bundle over $\mathcal{I}'_{2,0,3}$ and Φ is a projective bundle over Φ', applying this fact twice shows that the incidence correspondence is irreducible. The fibers over $\mathcal{I}'_{2,0,3}$ have dimension $4+12 = 16$. Because the base is 8-dimensional, Φ has dimension $16+8 = 24$.

Finally, the incidence correspondence Φ also projects onto the restricted Hilbert scheme $\mathcal{I}'_{4,1,3}$. Because any image of an irreducible scheme is irreducible, this shows that the restricted Hilbert scheme of curves of genus 4 and degree 1 has only one component. Riemann-Roch predicts that $C \in \mathcal{I}'_{4,1,3}$ lies on a $\mathbb{P}^1$ of quadrics and a $\mathbb{P}^7$ of cubics, and because C is a complete intersection, Corollary 3.3.3 implies that these are the actual dimensions. So the fibers of Φ over $\mathcal{I}'_{4,1,3}$ are 8-dimensional. Because Φ is 24-dimensional, the base $\mathcal{H}_{4,1,3}$ has dimension 16.

The upshot is that the dimension of a component of a Hilbert scheme can be computed from the dimension of a component of the Hilbert scheme parametrizing linked curves. This is a powerful technique for curves in $\mathbb{P}^3$. There are several difficulties, though. If the degree of a curve is large compared to the genus, then the curve often only needs to lie on surfaces of high degree. Such curves are not necessarily linked to another curve that is easier to describe.

Often, the difficult step is to count the number of surfaces of degree s and degree t containing the linked curve D. While Riemann-Roch predicts the number of surfaces containing D, curves need not lie on the expected number of surfaces when the degree of the surface is small. The expected dimension does give a lower bound on the actual dimension, which is sufficient in some but not all cases. Approaches to calculating the actual dimension vary, but one is to use linkage to relate cohomology groups.

The cohomology $H^1(\mathcal{I}_C(m))$ is trivial if and only if every section in $H^0(\mathcal{O}_C(m))$ is the restriction of a polynomial. To see this, consider the exact sequence

$$0 \to \mathcal{I}_C(m) \to \mathcal{O}_{\mathbb{P}^3}(m) \to \mathcal{O}_C(m) \to 0.$$

This gives a long exact sequence in cohomology:

$$\ldots \to H^0(C, \mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_C(m)) \to H^1(\mathbb{P}^3, \mathcal{I}_C(m)) \to H^1(C, \mathcal{O}_{\mathbb{P}^3}(m)) \to \ldots$$

and the final term on the right is zero. Therefore, computing this cohomology group can determine whether a curve lies on the expected number of surfaces of degree m.

Theorem 3.3.2. *Suppose C is residual to D in the intersection of surfaces S and T. Then $H^1(\mathcal{I}_{C/\mathbb{P}^3}(m))$ and $H^1(\mathcal{I}_{D/\mathbb{P}^3}(s+t-4-m))$ are dual vector spaces.*

Proof. Suppose that $C \cup D = S \cap T$. The cohomology of the curves can be related to the cohomology of surfaces. The sequence

$$0 \to \mathcal{I}_{S/\mathbb{P}^3}(m) \to \mathcal{I}_{C/\mathbb{P}^3}(m) \to \mathcal{I}_{C/S}(m) \to 0$$

is exact, and so gives a long exact sequence in cohomology. The first cohomology groups fit into the sequence

$$\ldots \to H^1(\mathcal{I}_{S/\mathbb{P}^3}(m)) \to H^1(\mathcal{I}_{C/\mathbb{P}^3}(m)) \to H^1(\mathcal{I}_{C/S}(m)) \to H^2(\mathcal{I}_{S/\mathbb{P}^3}(m)) \to \ldots.$$

Because $\mathcal{I}_{S/\mathbb{P}^3}(m) \cong \mathcal{O}_{\mathbb{P}^3}(m-s)$ has trivial first and second cohomology groups

$$h^1(\mathcal{O}_{\mathbb{P}^3}(m-s), \mathbb{P}^3) = h^2(\mathcal{O}_{\mathbb{P}^3}(m-s), \mathbb{P}^3) = 0,$$

this gives an isomorphism

$$H^1(\mathcal{I}_{C/\mathbb{P}^3}(m)) \cong H^1(\mathcal{I}_{C/S}(m)).$$

The right hand side is equal to $H^1(\mathcal{O}_S(m)(-C))$. Because the canonical line bundle ω_S on a surface S is isomorphic to $\mathcal{O}_S(s-4)$,

$$H^1(\mathcal{O}_S(m)(-C) = H^1(\omega_S(m-s+4)(-C)).$$

Because $\mathcal{O}_S(C+D) = \mathcal{O}_S(t)$, Serre duality shows the right hand side is equal to $H^1(\mathcal{O}_S(s+t-4-m)(-D))^*$ (where the asterisk indicates the dual vector space). Applying the same equalities to D in reverse,

$$H^1(\mathcal{I}_{C/\mathbb{P}^3}(m)) = H^1(\mathcal{I}_{D/\mathbb{P}^3}(s+t-4-m))^*.$$

This completes the proof. □

The derivation also shows that a complete intersection lies on the expected number of surfaces in each degree, a fact that was cited several times above. Because $H^1(\mathcal{I}_{C/\mathbb{P}^3}(m))$ measures the number of extra surfaces of degree m, it is sufficient to show this cohomology group vanishes for each m. Such a curve is called projectively normal.

Corollary 3.3.3. *If C is a complete intersection of curves S and T, then*

$$H^1(\mathcal{I}_{C/\mathbb{P}^3}(m)) = 0$$

for each $m \in \mathbb{Z}$. Equivalently, C lies on the number of surfaces of each degree predicted by Riemann-Roch.

Proof. The previous proof showed that $H^1(\mathcal{I}_{C/\mathbb{P}^3}(m)) = H^1(\mathcal{O}_S(m)(-C))$. In the case of a complete intersection, this is equal to $H^1(\mathcal{O}_S(m-t))$. But a surface in $\mathbb{P}^3$ has trivial first cohomology after twisting by any integer. □

The relationship between curves under linkage is determined by an algebraic invariant. Two curves C and D in $\mathbb{P}^3$ are linked if there exists a finite set of curves

$$C = C_0, C_1, ..., C_{n-1}, C_n = D$$

such that for each i, there exist surfaces S and T such that $C_i \cup C_{i+1} = S \cap T$. Identifying linked curves defines an equivalence relation called liaison on the set of curves in $\mathbb{P}^3$.

Definition 3.3.4. For a curve C, define the Hartshorne-Rao module

$$M_C = \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{I}_C(m), C).$$

Theorem 3.3.5. *(Hartshorne-Rao) Sending C to M_C gives a bijection between equivalence classes under liaison and finite graded modules over $\mathbb{C}[x_0, ..., x_3]$ up to twisting and dualizing. In particular, linked curves have the same Hartshorne-Rao modules.*

Theorem 3.3.2 implies that M_C modulo twisting and dualizing is a liaison invariant, so there is a well-defined map. Checking the map is a bijection is more difficult, but is shown in [12].

Chapter 4

Parametric Methods and Brill-Noether Theory

The methods of the previous chapter involved describing curves by using the geometry of the surfaces containing those curves. These methods are often very effective in low degree, and will be used repeatedly in Chapter 5. But as the degree of the curves grows, they are less effective, particularly if the genus is small. For example, the Picard group of a surface of degree $s \geq 4$ cannot be described as simply as the Picard groups of smooth quadric and cubic surfaces. When the dimension of the projective space $\mathbb{P}^r$ is more than three, describing restricted Hilbert schemes using the geometry of surfaces becomes even more difficult.

To see an example of how these methods break down even in three-dimensional projective space, consider the restricted Hilbert scheme $\mathcal{I}'_{10,6,3}$ of curves of degree 10 and genus 6. Riemann-Roch does not predict that such a curve must lie on any surfaces of degree 2, 3 or even 4, so there is no obvious description using the Picard group of a surface. Any curve C parametrized by a point in $\mathcal{I}'_{10,6,3}$ does need to lie on a pencil of quintic surfaces, but the residual curve has degree 15. By the genus-degree formula, the residual curve has genus 21. While the genus is now larger, Riemann-Roch does not show that these curves lie on surfaces of degree less than five either.

4.1 Rational Curves

An alternate approach is to describe a curve of given degree and genus parametrically. A point in the Hilbert scheme is determined uniquely by an abstract curve of the correct genus, a line bundle on that curve of the correct degree, and a choice of $r + 1$ sections of that line bundle.

Parts of this approach are exhibited in the case of the Hilbert scheme $\mathcal{H}_{d,0,r}$. The discussion of twisted cubics in Section 2.2 calculated the dimension of a component of the Hilbert scheme $\mathcal{H}_{3,0,3}$. The rational curves with d and r arbitrary are essentially the same.

Any irreducible rational curve $C \subset \mathbb{P}^r$ is the image of the projective line under a morphism $\mathbb{P}^1 \to \mathbb{P}^r$. If (x_0, x_1) are projective coordinates on $\mathbb{P}^1$, then the map is given by $(f_0(x_0, x_1), ..., f_r(x_0, x_1))$, where each f_i is a homogeneous polynomial of degree d. The incidence correspondence

$$\{(f_0(x_0, x_1), ..., f_r(x_0, x_1)) : f_i \text{ homogeneous polynomials of degree } d\}$$

is irreducible and has dimension $(r+1)(d+1)-1$. Moreover, an open subset of the incidence correspondence maps onto $\mathcal{H}_{d,0,r}$ by sending polynomials f_i to their zero set in $\mathbb{P}^r$. This immediately implies that $\mathcal{H}_{d,0,r}$ is irreducible. The fibers are three-dimensional, because two choices of polynomials f_i and g_i give the same curve precisely if f_i is given by precomposing g_i with an automorphism of $\mathbb{P}^1$. Subtracting the dimension of the fiber from the dimension of the incidence correspondence shows that $\mathcal{H}_{d,0,r}$ is irreducible of dimension $(r+1)(d+1)-4$.

To rephrase, any rational curve is isomorphic to $\mathbb{P}^1$ and the only line bundle of degree d on $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(d)$. Choosing $r+1$ homogeneous polynomials is the same as choosing $r+1$ sections of $\mathcal{O}_{\mathbb{P}^1}(d)$.

Can this method work for curves of positive genus? In general, Brill-Noether theory tries to describe arbitrary curves in this way by allowing the curve and line bundle to vary. The theory is most successful when the degree is relatively large, and especially when the Brill-Noether number

$$\rho(d, g, r) = g - (r+1)(d-g+r)$$

is positive. This chapter will highlight a few key results of the theory.

4.2 Definitions

Several definitions are helpful. The starting point is the moduli space $\mathcal{M}_g$ of abstract curves of genus g. Assume that the genus g is at least 2 so that the moduli space is irreducible of dimension $3g-3$ and generic curves are automorphism-free. Let $\mathcal{M}_g^0$ be the open subscheme of the moduli space parametrizing curves without non-trivial automorphisms. For a discussion of the geometry of the moduli space, see [4].

For any curve C in the moduli space $\mathcal{M}_g$, there is a g-dimensional scheme parametrizing line bundles of degree d on C known as the Jacobian $J_d(C)$. The Jacobian is often discussed in the theory of Abelian varieties. Over the locus $\mathcal{M}_g^0$ of automorphism-free curves, the Jacobians $J_d(C)$ fit together to give the Jacobian bundle $J_d \to \mathcal{M}_g$ parametrizing degree d line bundles.

The next two schemes describe special linear systems of a curve. Let $W_d^r(C) \subset J_d(C)$ be the locus parametrizing line bundles with at least $r+1$ linearly independent sections. These are the line bundles that give maps to the projective space $\mathbb{P}^r$. Similarly, let $G_d^r(C)$ be the scheme parametrizing linear systems g_d^r of degree d and dimension exactly r on C. There is a surjective map from $G_d^r(C) \to W_d^r(C)$ mapping a g_d^r to the line bundle $\mathcal{O}_C(D)$ for $D \in g_d^r$.

These schemes $W_d^r(C)$ and $G_d^r(C)$ fit together in families W_d^r and G_d^r over M_g^0. The scheme G_d^r is the object of interest, because each projective curve corresponds to an element of G_d^r along with a choice of a basis for the linear system g_d^r. In other words, the open subset of the Hilbert scheme parametrizing smooth curves is a $PGL_r(\mathbb{C})$ bundle over an open subset of G_d^r.

There is a series of maps such that an open subset of each scheme is a bundle over an open subset of the image:

$$
\begin{tikzcd}
\mathcal{I}'_{d,g,r} \arrow[d] \\
G_d^r \arrow[d] \\
W_d^r \subset J_d \arrow[d] \\
\mathcal{M}_g
\end{tikzcd}
$$

To give a straightforward example, suppose $d \geq 2g-1$. Then every line bundle of degree d is non-special, so every element of J_d is a line bundle with $d-g+1$ sections. Whenever $r \geq d-g+1$, Riemann-Roch gives an equality of schemes $W_d^r = J_d$. Moreover, because every scheme has $d-g+1$ sections, the fibers of G_d^r over W_d^r are each Grassmannians $G(r+1, d-g+1)$. Finally, the restricted Hilbert scheme has one component and its fiber over G_d^r have dimension $(r+1)^2-1$. Adding dimensions,

$$\dim \mathcal{I}'_{d,g,r} = 4g-3-(r+1)(d-g+1)-1$$

because J_d has dimension $4g-3$.

4.3 Expected Dimension

The varieties $W^r_d(C)$ and $G^r_d(C)$ can be described locally as determinantal schemes. Consider W^r_d near a point parametrizing a line bundle L. For n large enough, there are line bundles M of degree n and N of degree $n-d$ such that

$$L = M \otimes N^*,$$

where N^* is the dual of N.

Locally around L, there is an expression

$$L_\lambda = M \otimes N^*_\lambda.$$

The line bundle L can be vary with M fixed. Choose a divisor E such that $M = \mathcal{O}(E)$ and divisors D_λ such that $N = \mathcal{O}(D_\lambda)$, where

$$D_\lambda = \sum_{i=1}^{n-d} p_i(\lambda).$$

Moreover, take M and N_λ so that the $p_i(\lambda)$ are distinct and disjoint from the support of E.

Then if $f_1, ..., f_{n-g+1}$ are a basis for $H^0(M)$, each f_i is meromorphic with the only poles on the support of E. In particular, $f_i(p_j(\lambda))$ is finite for each i and j. The sections $H^0(L_\lambda)$ of L_λ are equal to the sections of $H^0(\mathcal{O}(E - D_\lambda))$, which are the meromorphic functions $f \in H^0(M)$ that vanish at each $p_i(\lambda)$.

So L_λ has $r+1$ sections whenever the co-rank of

$$\begin{pmatrix} f_1(p_1(\lambda)) & \dots & f_{n-g+1}(p_1(\lambda)) \\ \vdots & \ddots & \vdots \\ f_1(p_{n-d}(\lambda)) & \dots & f_{n-g+1}(p_{n-d}(\lambda)) \end{pmatrix}$$

is at most $r+1$, or the rank is at most $n-g-r$. This is a determinantal subscheme of $J_d(C)$, and taking the determinants of the suitable minors gives equations cutting out $W^r_d(C)$ near L.

For $G^r_d(C)$, the idea is to specify points as line bundles L_λ and subspaces in the Grassmannian $G(r+1, h^0(M))$ parametrizing $(r+1)$-dimensional subspaces of $H^0(M)$. In an open neighborhood around $g \in G(r+1, h^0(M))$, choose a basis

$$g_\mu = \{g_{1,\mu}, ..., g_{r+1,\mu}\}$$

for each subspace g_μ. Any section in $H^0(\mathcal{O}(L))$ must vanish at each $p_i\lambda$, so $G^r_d(C)$ is cut out by the local equations $g_{i,\mu}(p_j(\lambda)) = 0$ for each i and j.

These local equations give estimates for the dimension of $W^r_d(C)$, $G^r_d(C)$, and therefore $\mathcal{I}'_{d,g,r}$. In the space of $m \times n$ matrices, the determinantal subscheme of matrices with rank at most k has codimension $(m-k)(n-k)$. Therefore, the determinantal subscheme $W^r_d(C) \subset J_d(C)$ where

$$\begin{pmatrix} f_1(p_1(\lambda)) & \dots & f_{n-g+1}(p_1(\lambda)) \\ \vdots & \ddots & \vdots \\ f_1(p_{n-d}(\lambda)) & \dots & f_{n-g+1}(p_{n-d}(\lambda)) \end{pmatrix}$$

has rank at most $n-g-r$ has codimension bounded above by

$$((n-g+1)-(n-g-r))((n-d)-(n-g-r)) = (r+1)(g-d+r).$$

This follows from the theory of determinantal schemes, discussed in [13]. Because J_d has dimension g, the dimension of $W^r_d(C)$ is the minimum of g and

$$\rho(d,g,r) = g-(r+1)(g-d+r).$$

Recall this is the Brill-Noether number defined above.

If $g-d+r \geq 0$, then this shows that every component of $G^r_d(C)$ has dimension at least $\rho(d,g,r)$. If $g-d+r < 0$, then by Riemann-Roch, every line bundle $L \in J_d$ certainly has at least $d-g-r$ sections. So the Grassmannian of $(r+1)$-planes in $H^0(L)$ has dimension at least $(r+1)(d-g-r)$, which implies that $G^r_d(C)$ has dimension at least

$$\rho(d,g,r) = g+(r+1)(d-g-r).$$

Varying the curve C and choosing a basis for each subspace of dimension $r+1$, every component of $\mathcal{I}'_{d,g,r}$ has dimension at least

$$\rho(d,g,r) + 3g - 3 + (r+1)^2 - 1.$$

This is called the expected dimension of the component.

4.4 Results in the Brill-Noether Range

In the Brill-Noether range where $\rho(d,g,r) \geq 0$, much more has been shown. Several major results in this case are stated without proof.

The first question is whether $W^r_d(C)$ is non-empty. Whenever $\rho(d,g,r) \geq 0$, both $W^r_d(C)$ and $G^r_d(C)$ are non-empty. This can be shown by globalizing local equations similar to those given for the two varieties above. There is a proof due to Kempf, and another due to Kleiman and Laksov [14], [15]. As a corollary, the restricted Hilbert scheme has a component that dominates the moduli space.

The next result, due to Griffiths and Harris, is that $W^r_d(C)$ has dimension exactly $\rho(d,g,r)$ [16]. When $\rho(d,g,r)$ is negative, this implies that $W^r_d(C)$ is empty, so there is no component dominating the moduli space. When $\rho(d,g,r)$ is positive, this implies that the component dominating the moduli space has the expected dimension. Together, these two results give the Brill-Noether theorem: $W^r_d(C)$ is non-empty if and only if $\rho(d,g,r) \geq 0$.

Finally, Fulton and Lazarsfeld showed that $W^r_d(C)$ is connected whenever $\rho(d,g,r)$ is positive [17]. Combined with work of Geiseker, their result shows that $G^r_d(C)$ is irreducible [18].

Applying these results to the Hilbert scheme gives:

Theorem 4.4.1. *Suppose d, g and r are chosen so that $\rho(d,g,r) \geq 0$. Then there is a unique component of $\mathcal{I}'_{d,g,r}$ dominating the moduli space $\mathcal{M}_g$. Moreover, this component is irreducible and has the expected dimension*

$$\rho(d,g,r) + 3g - 3 + (r+1)^2 - 1.$$

The Brill-Noether component dominating the moduli space need not be the only component of the restricted Hilbert scheme, however, if d is not too large. For examples of extra components in the Brill-Noether range, see Section 5.3.

Chapter 5

Applications to Low Degree and Reducible Examples

5.1 Expected Dimension in Low Degree

Section 4.3 introduced the notion of the expected dimension, and showed that the expected dimension of a component of the Hilbert scheme gives a lower bound on the actual dimension. This section asks when the expected dimension is equal to the actual dimension.

In $\mathbb{P}^3$, the expected dimension of a component $\mathcal{H}_{d,g,3}$ is

$$\rho(d, g, 3) + 3g - 3 + (r + 1)^2 - 1 = 4d.$$

A plane curve of degree $d > 1$ has genus $\binom{d-1}{2}$. The corresponding Hilbert scheme has dimension

$$\binom{d+2}{2} - 1 + 3,$$

because the space of degree d plane curves in $\mathbb{P}^2$ has dimension $\binom{d+2}{2} - 1$ and each curve is contained in a unique plane. If $d = 2$ or $d = 3$, then $\dim \mathcal{H}_{d,g,3} = 4d$ is the expected dimension. In degree 4, however, the Hilbert scheme has dimension $17 > 4d$.

For an example where the curves are non-degenerate, the degree must be larger. Complete intersections, for example, can have dimension much larger than $4d$ as the degrees of the surfaces grow large.

The lowest degree example where a component of the restricted Hilbert scheme does not have the expected dimension is $d = 8$ and $g = 9$. Because curves of type $(4, 4)$ on a quadric surface have degree 8 and genus 9, these curves are dense in a component of

$\mathcal{I}'_{8,9,3}$. But the dimension of this component is

$$(4+1)(4+1)+8 = 33 > 4d.$$

Theorem 5.1.1. *If $d < 8$ or $d = 8$ and $g < 9$, then $\mathcal{I}'_{d,g,3}$ is irreducible and has the expected dimension $4d$. So $\mathcal{I}'_{8,9,3}$ is the lowest degree example of a restricted Hilbert scheme in $\mathbb{P}^3$ with dimension greater than $4d$.*

Degree Less Than 2g: If $d \geq 2g-1$, then the line bundle $\mathcal{O}(1)$ is non-special. To specify a curve $C \subset \mathbb{P}^3$ of degree d and g is equivalent to choosing an abstract curve of genus g, a line bundle in the Jacobian bundle, and four sections of that line bundle up to scalar multiplication. The total dimension is

$$3g - 3 + g + 4(d-g+1) - 1 = 4d,$$

as expected.

Castelnuovo Extremal Curves: Castelnuovo theory shows that the highest genus of curves of degree d is $(d/2-1)^2$ for d even and $(d/2-1/2)(d/2-3/2)$ for d odd. The Castelnuovo extremal curves are of type $(d/2, d/2)$ and type $((d-1)/2, (d+1)/2)$ on smooth quadric surfaces whenever $d \geq 7$. See the book by Arbarello, Cornalba, Griffiths and Harris for proofs of these claims [13].

If $d \leq 5$, then it is easy to check that the Castelnuovo extremal curves have genus satisfying $d \geq 2g-1$, so $\mathcal{I}'_{d,g,r}$ is irreducible of the expected dimension.

If $d = 6$, the Castelnuovo extremal curves have genus 4. By Riemann-Roch, any such curve must lie on a quadric. The family of curves of type $(3,3)$ has dimension

$$(3+1)(3+1)+8 = 24$$

as expected.

If $d = 7$, the facts above imply that the only component of the restricted Hilbert scheme $\mathcal{I}'_{7,6,3}$ is the family of curves of type $(3,4)$ on quadrics. This has dimension

$$(3+1)(4+1)+8 = 28,$$

as expected.

Note that in degree 8, the Castelnuovo extremal curves have degree 8 and genus 9.

The remaining cases to check are $d = 7$ and $g = 5$ as well as $d = 8$ and $g = 5, 6, 7$ or 8. The simplest cases are treated first.

Degree 8 and Genus 8: If $d = 8$ and $g = 8$, then by Riemann-Roch any curve must lie on a quadric surface. The curves are of type $(5, 3)$, and the corresponding Hilbert scheme has dimension 32.

Degree 7 and Genus 5: If $d = 7$ and $g = 5$, then by Riemann-Roch, any curve must lie on at least a $\mathbb{P}^2$ of cubic surfaces. By the degree-genus formula, the residual curve in the intersection of any two cubics has degree 2 and genus 0. The space of cubic surfaces containing a given conic has the expected dimension, so it is a $\mathbb{P}^{12}$. Because the space of conics in $\mathbb{P}^3$ is 8-dimensional, the incidence correspondence

$$\{(C, D, S, T) : C \text{ of degree 7 and genus } 5, D \text{ a conic}, S \text{ and } T \text{ cubic surfaces}\}$$

has dimension $8+12+12 = 32$. Because the fibers over $\mathcal{I}'_{7,5,3}$ are at least $4-$dimensional, the dimension of $\mathcal{I}'_{7,5,3}$ is at most 28. The dimension is always at least the expected dimension, so $\dim \mathcal{I}'_{7,5,3} = 28$.

Degree 8 and Genus 7: The case where $d = 8$ and $g = 7$ is similar. Here, Riemann-Roch shows any curve must lie on at least a pencil of cubic surfaces. The residual curve in their complete intersection is a line, which lies on a $\mathbb{P}^{15}$ of cubics. Since $4 + 15 + 15 = 34$, the incidence correspondence

$$\{(C, L, S, T) : C \text{ of degree 8 and genus } 7, L \text{ a line}, S \text{ and } T \text{ cubic surfaces}\}$$

is 34-dimensional. Because the fibers over $\mathcal{I}'_{8,7,3}$ are at least 2-dimensional, the restricted Hilbert scheme has the expected dimension of 32.

Degree 8 and Genus 6: If $d = 8$ and $g = 6$, then a curve C lies on at one cubic surface and at least a $\mathbb{P}^7$ of irreducible quartic surfaces. The curve C is residual to a rational quartic D in the intersection of any two cubic surfaces. A rational quartic must be a curve of type $(3, 1)$ on a quadric surface Q, which gives an exact sequence

$$0 \to \mathcal{O}(m-2) \to \mathcal{I}_D(m) \to \mathcal{O}_Q(m-3, m-1) \to 0.$$

Substituting the cohomology of $\mathcal{O}_{\mathbb{P}^1}(m)$ into the Kunneth formula

$$h^i(\mathcal{O}_Q(m-3, m-1)) = \sum_{i+j=k} h^i(\mathcal{O}_{\mathbb{P}^1}(m-3))h^j(\mathcal{O}_{\mathbb{P}^1}(m-1))$$

gives the cohomology groups $h^i(\mathcal{O}_Q(m-3, m-1))$. Applying the long exact sequence in cohomology gives the dimensions of the cohomology groups of $\mathcal{I}_D(m)$.

The cases $m = 2$ and $m = 3$ show that the ideal of D is generated by the quadric and three linearly independent cubics. So D lies on a $\mathbb{P}^6$ of cubics and a $\mathbb{P}^{17}$ of quadrics. The incidence correspondence

$$\{(C, D, S, T) : S \text{ a cubic surface and } T \text{ a quartic surface}\}$$

has dimension $16+17+6=39$. The fibers have dimension at least 7, so the dimension is equal to the expected dimension of 32.

Note that in each of these three cases where linkage was used, the incidence correspondence is irreducible because a projective bundle over an irreducible scheme is. Since the map from the incidence correspondence to the restricted Hilbert scheme is a dense morphism, the restricted Hilbert schemes are irreducible as well.

Degree 8 and Genus 5: The final case is $d=8$ and $g=5$, and will use more difficult theorems than the above. The first result needed is that if $d \geq g+3$, then a general line bundle is very ample [1]. Because a general line bundle is also non-special, there is an open subset of J_d where each line bundle has $d-g+1$ global sections.

Because J_d has dimension $4g-3$, this open subset gives a family of curves of dimension

$$4g-3+4(d-g+1)-1=4d$$

in the Hilbert scheme. The family is dense in a component of a Hilbert scheme, because no family of higher dimension dominates J_d. This is in fact the only component, because the restricted Hilbert scheme is irreducible. Irreducibility for $d=g+3$ in $\mathbb{P}^3$ has been shown by several authors, such as Keem and Kim [19].

5.2 Irreducibility in Low Degree

The previous section calculated the dimension of each component of a number of restricted Hilbert schemes in $\mathbb{P}^3$ and compared these dimensions to the expected dimension $4d$. In each case considered, there was only one component of the restricted Hilbert scheme. Can a restricted Hilbert scheme have multiple components?

For sufficiently large degree, the answer is no. In one paper, Keem and Kim show that $\mathcal{H}_{d,g,3}$ is irreducible if $d \geq g+3, d=g+2$ and $g \geq 5$, or $d=g+1$ and $g \geq 11$ [19]. Soon after, Hristo Iliev improved the bounds by proving irreducibility of the restricted Hilbert scheme for $d=g$ and $g \geq 13$ [20]. In the more general setting of $\mathbb{P}^r$, Harris showed that whenever

$$d > \frac{2r-1}{r+1}g+1,$$

$\mathcal{H}_{d,g,r}$ is irreducible [3]. When applied to $\mathbb{P}^3$, this bound is much weaker than the other results just cited.

But in general, the restricted Hilbert scheme can have multiple components. Consider curves of degree 9 and genus 10. Such a curve could be of type $(3,6)$ on a quadric surface, or could be the complete intersection of an intersection of cubic surfaces. By Riemann-Roch, any curve of degree 9 and genus 10 must lie on at least a $\mathbb{P}^1$ of cubic surfaces. If the pencil of cubics is irreducible, then the curve is a complete intersection. If the cubics are reducible, then the curve lies on a quadric. It must have type $(3, 6)$.

The family of curves of type $(3, 6)$ on a quadric has dimension $28+8 = 36$. The family of complete intersections of cubic surfaces has dimension $2\binom{6}{3} - 4 = 36$ as well. Because the two families have the same dimension and the families are distinct, neither family is contained in the closure of the other. The conclusion is that the restricted Hilbert scheme $\mathcal{H}_{9,10,3}$ has exactly two components and both have the expected dimension 36.

This is the lowest degree example of a reducible restricted Hilbert scheme in $\mathbb{P}^3$. Given the computations from Section 5.1, verifying this only requires checking a few more cases.

Theorem 5.2.1. *If $d < 9$ or $d = 9$ and $g < 10$, then the restricted Hilbert scheme $\mathcal{H}_{d,g,3}$ is irreducible. The Hilbert scheme $\mathcal{H}_{9,10,3}$ is the lowest degree reducible example.*

Degree Less Than 9: Each restricted Hilbert scheme in degree less than 9 is irreducible. This follows from the calculations in Section 5.1.

Degree 9 and Genus Less Than 6: If $g \leq 5$, then the line bundle $\mathcal{O}(1)$ is non-special. To specify a curve $C \subset \mathbb{P}^3$ of degree d and g is equivalent to choosing an abstract curve of genus g, a line bundle in the Jacobian bundle, and four sections of that line bundle up to scalar multiplication. Each choice gives a projective bundle over the previous scheme, so the restricted Hilbert scheme $\mathcal{H}_{9,g,3}$ is irreducible.

Degree 9 and Genus 6 and 7: These two cases follow from the result of Keem and Kim that $\mathcal{H}_{d,g,3}$ is irreducible whenever $d \geq g + 3$ or $d = g + 2$ and $g \geq 5$.

Degree 9 and Genus 8: Repeated application of linkage shows that $\mathcal{H}_{9,8,3}$ is irreducible. Suppose C is a curve of degree 9 and genus 8. Riemann-Roch shows that $h^0(C, \mathcal{O}_C(4)) = 29$, so C lies on a pencil of quartic surfaces. Because C lies on at most one cubic, a pencil of irreducible quartics can be chosen. In the intersection of the pencil, C is residual to a curve D of degree 7 and genus 4. To apply the techniques of linkage, the number of quartics containing D must be known. Another application of linkage will show that D lies on the expected number of quartics.

The curve D lies on the expected number of quartics if and only if the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_D(4))$$

is surjective. The exact sequence

$$0 \to \mathcal{I}_D(4) \to \mathcal{O}_{\mathbb{P}^3}(4) \to \mathcal{O}_D(4) \to 0$$

gives rise to an exact sequence.

$$0 \to H^0(\mathcal{I}_D(4)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_D(4)) \to H^1(\mathcal{I}_D(4)) \to 0$$

in cohomology. The conclusion is that D lies on the expected number of quartic surfaces precisely when $h^1(\mathcal{I}_D(4)) = 0$.

By Riemann-Roch again, any curve D of degree 7 and genus 4 must be contained in the complete intersection of cubic surfaces. The residual curve has degree 2 and genus -1, and this is generically a union of two skew lines $L \cup L'$. So the first cohomology groups of $\mathcal{I}_D(k)$ can be computed in terms of the first cohomology of $\mathcal{I}_{L\cup L'}(k)$. Recall from Theorem 3.3.2 of Section 3.3 that

$$h^1(\mathcal{I}_D(m)) = h^1(\mathcal{I}_{L\cup L'}(2-m))$$

for each m. If $m = 1$, then $h^1(\mathcal{I}_{L\cup L'}(2-m)) = 1$, because the regular functions on two skew lines must be constant on each line, which gives two degrees of freedom. For all other m, however, the first cohomology group is trivial, i.e. $h^1(\mathcal{I}_{L\cup L'}(2-m)) = 0$. To check this in the relevant case, note that $\mathcal{O}_{L\cup L'}(-2)$ has no global sections. By the long exact sequence in cohomology of

$$0 \to \mathcal{I}_{L\cup L'}(-2) \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{L\cup L'}(-2) \to 0,$$

this implies $h^1(\mathcal{I}_{L\cup L'}(-2)) = 0$. The conclusion is that $h^1(\mathcal{I}_D(4)) = 0$ as well, and D lies on the expected number of quartic surfaces.

With C and D as above, the incidence correspondence

$$\{(C, D, S, T) : S, T \text{ quartic surfaces}\}$$

is irreducible because it maps to an irreducible scheme with fibers a product of projective spaces. Because the projection to $\mathcal{H}_{9,8,3}$ is dense, the restricted Hilbert scheme of curves of degree 9 and genus 8 is irreducible as well.

Degree 9 and Genus 9: By Riemann-Roch, a curve C of degree 9 and genus 9 must lie on at least one cubic surface S and at least one irreducible quartic surface T. Linkage applies again. By the genus-degree formula, the curve C must be linked to a twisted cubic D.

The curve D is linked to a line L in the intersection of two quadric surfaces. Because $h^1(\mathcal{I}_L(m)) = 0$ for each m, it follows that $h^1(\mathcal{I}_D(m)) = 0$ for each m as well. Equivalently, a twisted cubic lies on the expected number of degree m surfaces for each m.

By Riemann-Roch, any twisted cubic D must lie on a $\mathbb{P}^9$ of cubic surfaces and a $\mathbb{P}^{21}$ of quartics. The restricted Hilbert scheme $\mathcal{H}_{3,0,3}$ has dimension 12, so the incidence correspondence

$$\{(C, D, S, T) : C \text{ a cubic and } D \text{ a quadric}\}$$

is irreducible of dimension 42. Because the projection to $\mathcal{H}_{9,9,3}$ is surjective, the restricted Hilbert scheme is irreducible. Because C lies on one cubic and at least a $\mathbb{P}^6$ of quartics, the dimension is equal to the expected dimension of 36.

5.3 Reducibility in the Brill-Noether Range

The example of a reducible restricted Hilbert scheme given in Section 5.2 has genus greater than the degree. It is not too difficult to produce more examples of reducible restricted Hilbert schemes with large genus. If the degree of the curves is increased sufficiently, though, various results show reducibility is impossible. One might ask whether the restricted Hilbert scheme is irreducible in the Brill-Noether range, where there is a unique component dominating the moduli space.

The answer is that the restricted Hilbert scheme can still have multiple components, at least when the projective space is enlarged from $\mathbb{P}^3$ to $\mathbb{P}^r$ for $r > 3$. The first example is due to Eisenbud and Harris [21], and was generalized by Keem [22].

The idea is to choose a subset of the moduli space and birationally very ample linear systems on the curves in that subset with as many sections as possible. Then any choice of $r+1$ linearly independent sections gives a curve of the desired degree and genus in $\mathbb{P}^r$. Hyperelliptic curves have linear systems with the most sections, but do not have birationally very ample special divisors. But the locus of trigonal curves and more generally k-gonal curves for $k \geq 3$ does have birationally very ample special divisors.

Consider the locus of trigonal curves. An abstract trigonal curve is determined by its branch locus up to an automorphism of $\mathbb{P}^1$. By the Riemann-Hurwitz formula, the branch locus has degree $|B| = 2g + 6 - 2 = 2g + 4$, so the trigonal locus has dimension $2g + 1$.

If K_C is the canonical linear system on each curve C and g^1_3 is the trigonal linear system, then $|K_C - mg^1_3|$ is birationally very ample for $m \leq \frac{g-4}{2}$. This follows from the fact that the canonical linear system K_C embeds C in a rational normal scroll in $\mathbb{P}^r$. The g^1_3 gives a ruling on the scroll, and $|K_C - g^1_3|$ embeds C in the rational normal scroll in $\mathbb{P}^{r-2}$ obtained by projecting along the ruling. In general, $|K_C - mg^1_3|$ embeds C in the rational normal scroll in $\mathbb{P}^{r-2m}$ obtained by projecting repeatedly along the ruling as long as $2m \leq g - 4$.

The linear system $|K_C - mg^1_3|$ has genus g and degree $d = 2g - 2 - 3m$. Suppose that m is chosen so that these curves has positive Brill-Noether number

$$\rho(d, g, r) = g - (r + 1)(g - d + r).$$

By Section 4.4, there is a unique irreducible component of $\mathcal{I}'_{d,g,r}$ dominating the moduli space. The Brill-Noether component has the expected dimension

$$\rho(d,r,g) + 3g - 3 + (r+1)^2 - 1 = d(r+1) - (r-3)(g-1).$$

When $d \geq 5$, the trigonal locus has positive codimension. Because the Brill-Noether component dominates the moduli space, it cannot be in the closure of any family of trigonal curves for $d \geq 5$. So it is sufficient to exhibit a family of trigonal curves with at least the expected dimension.

A choice of a trigonal curve gives $2g+1$ parameters. By Riemann-Roch, the linear system $|K_C - mg^1_3|$ has dimension

$$h^0(\mathcal{O}_C(K_C - mg^1_3)) = (2g-2-3m) - g + 1 + h^0(\mathcal{O}_C(mg^1_3)) = g - 2m.$$

The choices of $(r+1)$-dimensional subspaces are parametrized by the Grassmannian $G(r+1, g-2m)$, which has dimension $(r+1)(g-2m-r-1)$. Finally, a choice of four linearly independent in this subspace up to scalars gives $(r+1)^2 - 1$ more parameters. So the restricted Hilbert scheme is reducible if

$$2g + 1 + (r+1)(g-2m-r-1) + (r+1)^2 - 1 \geq d(r+1) - (r-3)(g-1),$$

where $d = 2g - 2 - 3m$. This simplifies to

$$(m+1)(r+1) \geq 2g - 4.$$

Moreover, for the curves to be in the Brill-Noether range and the linear systems to be birationally very ample,

$$\rho(d,g,r) = g - (r+1)(g-d+r) > 0 \text{ and } m \leq \frac{g-4}{2}.$$

These inequalities are actually satisfied simultaneously. Take $r = 5$ and $m = 3$, in which case $d = 2g - 11$. The Brill-Noether number is

$$\rho(2g-11, g, 5) = 7g - 96,$$

which is positive if $g > 13$. In this case, m is sufficiently small. On the other hand,

$$(m+1)(r+1) \geq 2g - 4$$

whenever $g < 15$. Taking $g = 14$ gives a restricted Hilbert scheme with multiple components in the Brill-Noether range. This is one of many examples of choices of g, d, and r with these properties.

While trigonal curves can be thought of as having the most linear systems, curves of higher gonality exhibit similar behavior. The geometry becomes more complicated, but the example above is essentially typical. Keem showed that if $m \leq \lfloor \frac{g}{k-1} \rfloor - 2$ and C is k-gonal, then $|K_C - mg^1_k|$ is birationally very ample. Keem concludes the following theorem [22]:

Theorem 5.3.1. *Choose g, k, m, and r such that $k \geq 3$, $r \geq 3$, $2 \leq m \leq \lfloor \frac{g}{k-1} \rfloor - 2$, and*

$$\frac{2g + 2 - 2k}{m + 1} - 1 < r \leq g - 2 - mk.$$

Then taking $d = 2g - 2 - mk$, the restricted Hilbert scheme $\mathcal{I}'_{d,g,r}$ is reducible with a component dominating the moduli space and a distinct component containing each k-gonal curve.

These inequalities are actually satisfied for a large family of k, r and m, particularly as r grows larger.

Chapter 6

Bibliography

[1] Robin Hartshorne, *Algebraic Geometry.* Springer, 1977.

[2] David Eisenbud and Joe Harris, *The Geometry of Schemes.* Springer, 2000.

[3] Joe Harris, *Curves in Projective Space.* Presses de l'Universit de Montral, 1982.

[4] Joe Harris and Ian Morrison, *Moduli of Curves.* Springer, 1998.

[5] Robin Hartshorne, *Deformation Theory.* Springer, 2012.

[6] R. Hartshorne, "Connectedness of the Hilbert Scheme," *Publications mathématiques de l'I.H.É.S.*, vol. 29, pp. 5–48, 1966.

[7] A. Grothendieck, "Techniques de construction et théorèmes d'existence en géométrie algébrique IV: les schémas de Hilbert," *Séminaire Bourbaki*, vol. 6 Expose 221, p. 28, 1960-61.

[8] D. Mumford, "Further Pathologies in Algebraic Geometry," *American Journal of Mathematics*, vol. 84, no. 4, pp. 642–648, 1962.

[9] David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry.* Springer, 2004.

[10] David Eisenbud and Joe Harris, *3264* & *All That: Intersection Theory in Algebraic Geometry.* In preparation, to appear.

[11] Arnaud Beauville, *Complex Algebraic Surfaces.* Cambridge University Press, 1996.

[12] A. P. Rao, "Liaison among curves in $\mathbb{P}^3$," *Inventiones mathematicae*, vol. 50, no. 3, pp. 205–217, 1978-79.

[13] Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths and Joe Harris, *Geometry of Algebraic Curves: Volume I.* Springer, 1985.

[14] D. Klempf, *Schubert methods with an application to algebraic curves.* Mathematisch Centrum, 1971.

[15] S. Kleiman and D. Laksov, "On the existence of special divisors," *American Journal of Mathematics*, vol. 94, no. 2, pp. 271–283, 1972.

[16] P. Griffiths and J. Harris, "On the variety of special linear systems on a general algebraic curve," *Duke Mathematical Journal*, vol. 47, no. 1, pp. 233–272, 1980.

[17] W. Fulton and R. Lazarsfeld, "On the connectedness of degeneracy loci and special divisors," *Acta Mathematica*, vol. 146, no. 1, pp. 271–283, 1981.

[18] D. Geiseker, "Stable curves and special divisors: Petri's conjecture," *Inventiones mathematicae*, vol. 66, no. 2, pp. 431–436, 1982.

[19] C. Keem and S. Kim, "Irreducibility of a Subscheme of the Hilbert Scheme of Complex Space Curves," *Journal of Algebra*, vol. 145, pp. 240–248, 1992.

[20] H. Iliev, "On the irreducibility of the Hilbert scheme of space curves," *Proceedings of the American Mathematical Society*, vol. 134, pp. 2823–2832, 2006.

[21] D. Eisenbud and J. Harris, "Irreducibility of some families of linear series with Brill-Noether number -1," *Annales scientifiques de l'Ecole Normale Superieure*, vol. 22, no. 1, pp. 33–53, 1989.

[22] C. Keem, "Reducible Hilbert Scheme of Smooth Curves with Positive Brill-Noether Number," *Proceedings of the American Mathematical Society*, vol. 122, no. 2, pp. 349–354, 1994.

[23] Enrico Arbarello, Maurizio Cornalba and Phillip Griffiths, *Geometry of Algebraic Curves: Volume II.* Springer, 2011.