# Defining physics at imaginary time: reflection positivity for certain Riemannian manifolds

A thesis presented

by

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to

the Department of Mathematics

in partial fulfillment of the requirements for an honors degree.

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Harvard University

Cambridge, Massachusetts

March 2013

# Contents

# 1 Introduction

Two concepts dominate contemporary physics: relativity and quantum mechanics. They unite to describe the physics of interacting particles, which live in relativistic spacetime while exhibiting quantum behavior. A putative theory of particles is referred to as a quantum field theory (QFT), and the most famous example is Yang-Mills theory, which is the basis of the Standard Model and accurately predicts the behavior of all observed particles. A QFT is a calculational framework, and efforts have been made to deduce the essential features of QFTs and express them as mathematical axioms. The Wightman axioms, formulated in the 1950s, provide the most prominent example [38]. Although satisfactory axiom schemes have been known for sixty years, it remains an open problem to construct an axiomatic version of Yang-Mills or any other physically plausible QFT [26]. Mathematical quantum field theories have been constructed for two and three spacetime dimensions, but the case of four spacetime dimensions (our physical universe) is still open. This is not the first moment in history when physics outpaced the ability of mathematics to describe it, and such moments herald growth for both fields. The successful constructions in two and three dimensions, as well as the current efforts in four dimensions, rely on constructing a theory of Euclidean fields and analytically continuing it to imaginary time. This analytic continuation is the topic of the present paper.

A mathematical QFT is based in the static Lorentzian manifold that is spacetime. The time coordinate is distinguished from the local spatial coordinates by a minus sign in the metric, and the condition that the manifold is static means that time-translation on the manifold is well-defined. The QFT is then built around a unitary representation of the isometry group of this manifold. In the prototypical case that the manifold is flat Minkowski space the metric is

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2,$$

and the isometry group is the Poincaré group. For a general spacetime, if we pass to imaginary time $t \mapsto it$, then the Lorentzian signature becomes Euclidean, and we achieve a new manifold with a Euclidean symmetry group. Arthur Wightman and others understood in the early 1950s that any QFT can be analytically continued in this way to a theory of Euclidean fields, meaning a theory of generalized functions with Euclidean symmetries. Passing to imaginary time is now a staple of the study of physicists' QFTs and incites traditional suspicion from first-year students of the topic.

This suggests the question: can we go in the other direction? Given a theory of Euclidean fields, what conditions are sufficient to analytically continue it to a mathematical QFT? Edward Nelson gave one set of conditions in the early 1970s that has not been checked for non-trivial examples [32]. Konrad Osterwalder and Robert Schrader made the breakthrough in 1973 when they discovered *reflection positivity*, which makes analytic continuation possible in this context and in many others [34, 35]. Osterwalder and Schrader showed that any Euclidean field theory equipped with a reflection positive bilinear form can be analytically continued to a mathematical QFT. They proved that the Wightman axioms follow from reflection positivity along with other conditions. This approach has been used to construct mathematical QFTs in two [17] and three [8, 14] spacetime dimensions.

In the present paper, we define reflection positivity and use it to give the following construction: Given a static Riemannian manifold $M$ with isometry group $\mathrm{Isom}(M)$, there is a related Lorentzian manifold $M_{\mathrm{lor}}$ with isometry group $\mathrm{Isom}(M_{\mathrm{lor}})$. Under certain physically plausible conditions on $M$, we show how to construct a reflection positive, unitary representation of $\mathrm{Isom}(M)$ that can be analytically continued to a unitary representation of $\mathrm{Isom}(M_{\mathrm{lor}})$. In addition to being mathematically interesting, this construction is at the heart of quantum field theory, and we show how it relates to the Wightman axioms for a non-interacting bosonic particle. In support of this construction, we construct various reflection positive bilinear forms on spaces over $M$.

We keep our treatment of reflection positivity abstract enough that its broad utility can be appreciated. The introductory discussion in Sections 3 and 4 encompasses the usage of reflection positivity in contexts with no immediate connection to mathematical QFT. These include the use of reflection positivity by Arthur Jaffe et al. to construct representations of the Heisenberg algebra on a Riemann surface [22], and the use of a reflection positive form by Vasily Pestun to compute the partition function for supersymmetric Yang-Mills on the four-sphere [36].[1]

## 2 Axiomatic quantum field theory

Physics is set in a static $d$-dimensional Lorentzian manifold $M_{\mathrm{lor}}$ called spacetime. The prototypical example is $d$-dimensional Minkowski space, which is the setting for Einstein's special relativity. In general, the case $d = 1$ corresponds to non-relativistic quantum mechanics, and the case $d = 4$ is believed to correspond to the physical universe. For purposes of quantum field theory, it must be possible to decompose the spacetime manifold as $\mathbb{R} \times \Sigma$ where $\mathbb{R}$ is the time axis and the hypersurface $\Sigma$ is a spatial cross-section. This decomposition is necessary because time-evolution of quantum states is fundamental to the development of a QFT.

Quantum field theories describe particles in this spacetime. The central object of the mathematical theory is a Hilbert space $\mathcal{H}$ of physical states equipped with a continuous unitary representation $U$ of the isometry group of spacetime $\mathrm{Isom}(M_{\mathrm{lor}})$. In the case that $M_{\mathrm{lor}}$ is Minkowski space, this means that $\mathcal{H}$ is equipped with a representation of the Poincareé group.

[1]It is an original observation of the author that the bilinear form defined by equations 3.2 – 3.4 of the cited paper is reflection positive. The reflection, denoted as conjugation, is induced by a reflection on $S^4$.

In addition to the representation $U$, the Hilbert space is equipped with a field operator $\phi$. The field operator is an operator-valued distribution, so it is a map from some space of test functions to $\mathrm{End}(\mathcal{H})$. Now and in the rest of the paper, this space of test functions will be denoted $\mathscr{H}(M_{\mathrm{lor}})$ with a continuous dual $\mathscr{H}'(M_{\mathrm{lor}})$ containing the distributions. The test functions are some refinement of $L^2(M_{\mathrm{lor}})$. In the case that $M_{\mathrm{lor}}$ is Minkowski space, it is common to take $\mathscr{H} = \mathcal{S}$, Schwartz space. A discussion of test function spaces is presented in 4.1.

Quantum field theory is defined by requiring that the the Hilbert space $\mathcal{H}$, representation $U$, and field operator $\phi$ satisfy the axioms of Arthur Wightman [16, 21, 38]. These axioms were originally formulated only in the case that $M_{\mathrm{lor}}$ is Minkowski space, but we have broadened their phrasing to apply to the more general setting of this paper:

**Axioms 2.1** (Wightman).

1. *The representation $U$ is positive-energy, which means that the generator $p_0$ of the representation of time translation satisfies $p_0 \geq 0$.*
2. *There exists an invariant vacuum vector $\Omega = U\Omega \in \mathcal{H}$.*
3. *The field $\phi$ transforms covariantly under $U$. This means that for $g \in \mathrm{Isom}(M_{\mathrm{lor}})$, we have $U(g)\phi(f)U(g)^\dagger = \phi(g_* f)$ where $g_* f = f \circ g^{-1}$.*
4. *Vectors of the form $\phi(f_1)\ldots\phi(f_n)\Omega$ for $f_j \in \mathscr{H}$ and arbitrary $n$ span $\mathcal{H}$.*
5. *The quantum field $\phi$ is local, meaning that if $f$ and $g$ have space-like separated support then $\phi(f)\phi(g) = \pm\phi(g)\phi(f)$.*
6. *The vacuum vector is the unique vector (up to scalar multiplication) in $\mathcal{H}$ that is invariant under time translation.*

The success of Osterwalder and Schrader was to formulate equivalent axioms for a theory of generalized functions with Euclidean symmetries. They constructed a Hilbert space $\mathcal{E}$ equipped with a unitary representation of the Euclidean group (the symmetries of $\mathbb{R}^d$) and with an operator-valued distribution $\Phi$. In the Euclidean picture, $\Phi(f_1)$ and $\Phi(f_2)$ are commuting or anti-commuting operators for any choice of $f_1, f_2$. This is in contrast to the quantum field $\phi$, which has non-trivial commutation relations. In [34, 35], Osterwalder and Schrader give conditions such that their Euclidean formulation is equivalent to the Wightman formulation. The crucial condition is reflection positivity. In recent papers by A. Jaffe and G. Ritter, much of the original construction is generalized to the setting where $\mathbb{R}^d$ is replaced with a static Riemannian manifold $M$ [23, 24, 25]. This is the point of view that we follow in this exposition.

The commutation relations for the Euclidean field $\Phi$ determine whether it is a bosonic particle or fermionic particle. The developments of Osterwalder and Schrader apply to both cases, but only the bosonic case is considered here. In the Osterwalder and Schrader picture for bosonic particles, we begin with a space of test functions $\mathscr{H}(M)$ and construct a Borel probability measure $d\mu(\Phi)$ on its continuous dual $\mathscr{H}'(M)$. Developing the theory of possible measures has been one of the major contributions of constructive quantum field theory to mathematics. In this paper, we will study the prototypical example of Gaussian measures. The measure gives the Euclidean field and will be used to produce the quantum field $\phi$. A natural representation of Isom$(M)$ on $\mathscr{H}(M)$ will be used to produce $\mathcal{H}$ and $U$. Osterwalder and Schrader gave axioms for $d\mu$ such that this Euclidean theory is equivalent to a Wightman quantum theory. In outline, these are:

**Axioms 2.2** (Osterwalder-Schrader)**.** *Properties of* $d\mu(\Phi)$*:*

1. *A regularity condition.*
2. *A clustering condition.*
3. *Euclidean covariance.*
4. *Reflection positivity.*

These axioms will be discussed as needed in Section 4. Axioms 3 and 4, which are critical to our exposition, are detailed as Axiom 4.4 and Axiom 4.12. For the proof of equivalence with the Wightman axioms, which is not proven in the present exposition, see [16, 34, 35].

# 3 Definition of reflection positivity

In this section, we define reflection positivity by giving two examples that build up to Definition 3.4.

Reflection positivity is a condition for a space $\mathcal{E}$ equipped with a bilinear form $\langle\cdot,\cdot\rangle$. In the presence of reflection positivity, the structure of $\mathcal{E}$ can be analytically continued to a new Hilbert space $\mathcal{H}$ with inner product a continuation of $\langle\cdot,\cdot\rangle$. Our first example is the following proposition due to Fitzgerald [9]:

**Proposition 3.1.** *Assume* $f(z_1, z_2)$ *is analytic for* $|z_1| < 1$ *and* $|z_2| < 1$*, and choose* $0 < \varepsilon < 1$*. If for any sequence of real numbers* $s_1, \ldots, s_m$ *in* $[0, \varepsilon]$ *the matrix* $a_{jk} = f(s_j, s_k)$ *is positive definite then for any sequence of real numbers* $r_1, \ldots, r_n$ *in* $[0, 1]$*, the matrix* $b_{jk} = f(i \cdot r_j, -i \cdot r_k)$ *is positive definite.*

*Proof.* Fitzgerald proves that the positivity of $a_{jk}$ leads to the positivity of $b_{jk} = f(z_j, \bar{z}_k)$ for any choice of $z_i$ in the unit circle. Here the bar denotes complex conjugation. □

This proposition is essentially the converse of a reflection positivity argument: We begin with a function $f$ that can be analytically continued. At the cost of a minus sign, we are able to analytically continue a positivity condition for real parameters to a positivity condition for imaginary parameters. In the setting of this paper, the positivity condition for imaginary parameters will imply the existence of analytic continuation, and the minus sign will be replaced with a more general involution referred to as "reflection."

The following example due to A. Uhlmann [41] illustrates how to define reflection-positivity in a setting with a more complicated reflection. Consider the infinite-dimensional real vector space $\mathcal{E}$ of distributions given by

$$f(\vec{r}) = \sum_{j=1}^{N} q_j \delta(\vec{r} - \vec{r}_j). \tag{1}$$

A physicist can think of these as charge distributions with finite self-energy. Define the following bilinear form on this space:

$$\langle f', f \rangle = \int d^3\vec{r}_1 d^3\vec{r}_2 \frac{f'(\vec{r}_1) f(\vec{r}_2)}{\|\vec{r}_1 - \vec{r}_2\|}.$$

This form gives the interaction energy of charge distributions. Define a unitary involution $\Theta$ on $\mathcal{E}$ as the pullback of a reflection on $\mathbb{R}^3$:

$$(\Theta f)(x, y, z) = f(-x, y, z).$$

We will refer to $\Theta$ as a reflection on $\mathcal{E}$.

**Proposition 3.2.** *Let* $\mathcal{E}_+ = \{f \in \mathcal{E} : f(x, y, z) = 0 \text{ if } x \leq 0\}$. *Then*

$$b(f, f) \equiv \langle f, \Theta f \rangle \geq 0 \quad \text{for all } f \in \mathcal{E}_+.$$

*Proof.* Suppose that $f \in \mathcal{E}_+$. Then we have that

$$f(\vec{r}) = \sum_{j=1}^{N} q_j \delta(\vec{r} - \vec{r}_j),$$

where the first component of $\vec{r}_j$ is greater than or equal to zero. We write the components of $\vec{r}$ as $x, y, z$, so this is the assertion that $x_j \geq 0$ for all $j$. Now we wish to prove:

$$\sum_{\ell, m} \frac{q_\ell q_m}{\|\vec{r}_\ell - \vec{r}_m\|} \geq 0.$$

Fourier transforming $1/r$ to $1/k^2$ and writing

$$a_\ell = q_\ell \exp[i(k_y y_\ell + k_z z_\ell)]$$

lets us rewrite this inequality as

$$\int \sum_{\ell,m} a_\ell \bar{a}_m k^{-2} \exp[i(x_\ell + x_n)k_x] d^3k \geq 0.$$

Let $c(k_y, k_z) = \sqrt{k_y^2 + k_z^2}$. Then performing the integration with respect to $k_x$ gives the left-hand side as

$$\sum_{\ell,m} \int a_m \bar{a}_\ell \frac{\pi \exp[c^{-1}(x_\ell + x_m)]}{c} dk_y dk_z$$

which is greater or equal to zero. □

This shows that $b(\cdot,\cdot)$ is a semi-positive bilinear form on $\mathcal{E}_+$. Let $N$ denote the space of vectors that have norm zero with respect to $b$. Then the following lemma allows us to construct a Hilbert space as the completion of $\mathcal{E}_+/\mathcal{N}$:

**Lemma 3.3.** *Let $b$ be a semi-positive bilinear form on a vector space $\mathcal{E}_+$, and let $\mathcal{N}$ be the null space of $b$, then $\mathcal{N}$ is a linear space and $b$ gives a positive-definite bilinear form on $\mathcal{E}_+/N$.*

*Proof.* We will show that for $u, v \in \mathcal{E}_+$ and $N \in \mathcal{N}$, $b(u+N, v) = b(u, v)$. This follows from the fact that the Cauchy-Schwarz inequality holds for $b$:

$$|b(u,v)| \leq b(u,u)^{1/2} b(v,v)^{1/2}.$$

□

The above discussion illustrates the following characteristic properties of reflection positivity:

**Definition 3.4** (Reflection positivity)**.** *Reflection positivity constructions have the following components:*

1. *$\mathcal{E}$: A real or complex vector space $\mathcal{E}$ equipped with a Hermitian form $\langle\cdot,\cdot\rangle$. Although this form was positive definite in the previous example, this need not be the case.*

2. ***$\Theta$**: An operator $\Theta$ on $\mathcal{E}$ such that $\Theta^2 = \mathrm{id}$ and $\langle \Theta f, g\rangle = \langle f, \Theta g\rangle$.*

3. *$\mathcal{E}_+$ : A linear subspace $\mathcal{E}_+ \subset \mathcal{E}$ with the property that for all $f \in \mathcal{E}_+$,*

$$\langle f, \Theta f\rangle \geq 0.$$

*These conditions mean that $\langle\cdot,\cdot\rangle$ is a reflection positive form on $\mathcal{E}$. We can then construct a related Hilbert space $\mathcal{H}$ by beginning with $\mathcal{E}_+$ and modding out by the nullspace of $b(\cdot,\cdot) = \langle\cdot, \Theta\,\cdot\rangle$. The art of reflection positivity is to construct additional structures on $\mathcal{E}$ that survive the construction of $\mathcal{H}$. We use this notation consistently throughout the paper.*

# 4 Reflection positivity on a Riemannian manifold $M$

In the case of quantum field theory, we will equip $\mathcal{E}$ with a unitary representation of a Euclidean symmetry group and with a Euclidean field. These structures will continue to exist on $\mathcal{H}$, and the construction of $\mathcal{H}$ will cause them to be analytically continued to quantum structures. To achieve this structure on $\mathcal{E}$, we will choose it to be a particular function space over $M$.

For the remainder of the present paper, unless stated otherwise, let $M$ denote a complete, connected, Riemannian manifold. For the purposes of quantum field theory, we further demand that $M$ be static in the sense of the following definition:

**Definition 4.1** (Static manifold). *A manifold with metric $(M, g_{\mu\nu})$ is static if it possesses a globally defined, hypersurface orthogonal Killing field $\xi$. Physicists refer to this as the generator of time translation, and the global coordinate along this field is written $t$. Then the manifold $M$ can be decomposed as $M = \mathbb{R} \times \Sigma$. The metric can be written locally as*

$$ds^2 = F(x)dt^2 + \sum_{i,j=1}^{\dim \Sigma} G_{ij}(x)dx^i dx^j$$

*where $F$ and $G$ depend only on the $\Sigma$ coordinates.*

*Any static Riemannian manifold $M$ is embedded in a complex manifold with Euclidean section (see [18, 7] for discussion) and has a Lorentzian continuation $M_{\mathrm{lor}}$ with a metric that can be written locally as*

$$ds^2 = -F(x)dt^2 + \sum_{i,j=1}^{\dim \Sigma} G_{ij}(x)dx^i dx^j.$$

It makes sense to discuss time evolution on such a manifold, so it is a natural setting for physics. Examples of static manifolds include Minkowski space, de Sitter space, and anti de Sitter space. The fact that a static manifold has an analytic continuation means that it is an ideal setting for defining mathematical quantum field theory. We will use reflection positivity to analytically continue a representation of $\mathrm{Isom}(M)$ to a representation of the identity component of $\mathrm{Isom}(M_{\mathrm{lor}})$.

## 4.1 Function space $\mathcal{E}$ over $M$

Let $M$ be a static Riemannian manifold. The vector space $C_0^\infty(M)$ consists of the smooth functions of compact support on $M$, and the vector space $L^2(M)$ consists of functions integrable under the Riemannian volume form. In the present paper, we will take both spaces to be real. These spaces of functions can be completed under various norms to give important function spaces on

$M$. For quantum field theory, we construct a space $\mathscr{H}(M)$ of test functions such that its continuous dual $\mathscr{H}'(M)$ is equipped with a Borel probability measure. This is easiest to accomplish in the case that $\mathscr{H}$ is a nuclear space. Nuclear spaces will be defined and discussed in section 7.1, but an example is:

**Example 4.2** (Schwartz space)**.** ***Schwartz space*** *for the manifold $M = \mathbb{R}^d$ with the standard metric is denoted $\mathcal{S}(\mathbb{R}^d)$. It is the space of functions $f \in C^\infty$ such that for all $\alpha, \beta$ we have $\|f\|_{\alpha,\beta} < \infty$ where*

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbf{R}^n} \left| x^\alpha D^\beta f(x) \right| .$$

*Schwartz space is a nuclear space. The continuous dual of Schwartz space, called the* ***tempered distributions****, is denoted $\mathcal{S}'(\mathbb{R}^d)$.*

*Schwartz space can, in principle, be defined in any setting where there is a notion of decay at infinity. An example of such generalized Schwartz functions is Harish-Chandra's Schwartz space over a semisimple Lie group [19, Section 9].*

In 7.1, we give an original construction of a nuclear space $\mathscr{H}(M)$ over any manifold $M$ where the spatial hypersurface $\Sigma$ is compact. The papers [23, 24] assert that a convenient choice of nuclear space exists for any static Riemannian manifold $M$, but there is an error in their construction.[2] The question of how to construct a suitable space of test functions for an arbitrary spacetime $M$ remains open.

The nuclear space of test functions $\mathscr{H}(M)$ will contain $L^2(M)$ and $C_0^\infty$ as dense sets. It is a refinement of these spaces in order to better accommodate distributions. We will describe the elements of $\mathscr{H}(M)$ as functions and refer to properties such as "support" that are only defined for true functions. These ideas can be extended to $\mathscr{H}(M)$ from $C_0^\infty(M)$. Throughout the current paper, all spaces of test functions are taken to be real.

Given a function space $\mathscr{H}$ and its continuous dual $\mathscr{H}'$, we write $\Phi(f)$ for the pairing of $f \in \mathscr{H}$ and $\Phi \in \mathscr{H}'$. Alternatively, we write this in terms of an integral kernel:

$$\Phi(f) = \langle f, \Phi \rangle = \int_M \Phi(x) f(x) dx \in \mathbb{R}.$$

Given the measure $d\mu(\Phi)$ on $\mathscr{H}'$, we can integrate functions in $\mathscr{H}$ as

$$\int_M \Phi(f) d\mu(\Phi).$$

The Bochner-Minlos theorem (see Appendix A) asserts that $d\mu$ is equivalent to its generating function:

$$S\{f\} = \int e^{i\Phi(f)} d\mu.$$

[2]The papers assume that the embedding of Sobolev spaces is Hilbert-Schmidt for any manifold $M$. The error was reported to the authors in March 2013.

We will follow J. Fröhlich [10] in using the generating function to state the important properties of $d\mu$.

We define our Euclidean Hilbert space as the complex vector space

$$\mathcal{E} \equiv L^2(\mathscr{H}', d\mu).$$

We write $P(\Phi) = \Phi(f_1)\dots\Phi(f_n)$ for the monomial in $\mathcal{E}$ given by the finite sequence of functions $f_j \in \mathscr{H}$. Given certain regularity conditions on $d\mu$, the complex span of these monomials is a dense set in $\mathcal{E}$. The space $\mathcal{E}$ then carries an action of $\mathrm{Isom}(M)$ in the following way:

**Definition 4.3** (Induced operator $\Gamma(\psi)$)**.** *Let $\psi \in \mathrm{Isom}(M)$. This induces an action on $C_0^\infty(M)$ by $\psi_* f = (\psi^{-1})^* f = f \circ \psi^{-1}$. Let $P(\Phi) = \Phi(f_1)\cdots\Phi(f_n) \in \mathcal{E}$ be a monomial. Then we define*

$$\Gamma(\psi)P \equiv \Phi(\psi_* f_1)\cdots\Phi(\psi_* f_n).$$

*This operator $\Gamma(\psi)$ extends linearly to the dense domain of polynomials in $\mathcal{E}$.*

This defines a group representation for $\mathrm{Isom}(M)$, and if $\psi_1, \psi_2 \in \mathrm{Isom}(M)$ commute then $[\Gamma(\psi_1), \Gamma(\psi_2)] = 0$. For quantum field theory, we demand that the measure $d\mu$ be such that this representation is unitary:

**Axiom 4.4** (Euclidean invariance of measure)**.** *This is one of the Osterwalder-Schrader axioms, see Axioms 2.2. We demand that $d\mu$ be such that $\Gamma(\mathrm{Isom}(M))$ is a strongly continuous unitary representation. This is accomplished when the generating function of the measure satisfies*

$$S\{f\} = S\{\psi_* f\},$$

*for all $\psi \in \mathrm{Isom}(M)$. Section 7 gives a recipe for constructing such measures.*

In addition to this action of $\mathrm{Isom}(M)$, $\mathcal{E}$ also carries a Euclidean field operator that, via analytic continuation, will give the quantum field operator of the Wightman axioms:

**Definition 4.5** (Euclidean field operator)**.** *Consider the operator-valued distribution on $\mathcal{E}$ defined in the following way on monomials: Given $f \in \mathscr{H}$, the operator-valued distribution gives the operator*

$$\Phi(f_1)\cdots\Phi(f_n) \mapsto \Phi(f)\Phi(f_1)\cdots\Phi(f_n).$$

*This extends by linearity to a densely defined operator on $\mathcal{E}$. In fact, polynomials in this field operator give a densely defined set in $\mathcal{E}$ by acting on the constant function* **1**. *Note that these operators commute for all $f$. This is the heart of the Euclidean field theory for bosonic particles: it is a theory of commuting fields.*

In sections 5 – 8, we use reflection positivity to construct a quantum Hilbert space $\mathcal{H}$ from $\mathcal{E}$. Operators on $\mathcal{E}$ that satisfy suitable conditions can be "quantized" to give operators on $\mathcal{H}$ (see section 5.1). The Euclidean field operator will quantize to give the quantum field. The action of the Euclidean group $\mathrm{Isom}(M)$ will quantize to give a strongly continuous unitary representation of the identity component of $\mathrm{Isom}(M_{\mathrm{lor}})$ on $\mathcal{H}$.

For the constructions in the following sections, we use the following notion of domains in $\mathcal{E}$: For any open set $O \subset M$, the corresponding domain in $\mathcal{E}$ is denoted $\mathcal{E}_O$, and it is defined as the closure of

$$E_O = \mathrm{span}\left\{e^{i\Phi(f)} : f \in \mathscr{H}(M),\ \ \mathrm{supp}(f) \subset O\right\}.$$

We have:

**Remark 4.6.** *Let $\psi \in \mathrm{Isom}(M)$, and suppose $\psi|_N : N \to O$ where $N, O$ are open sets in $M$. Let $\mathcal{E}_N$ and $\mathcal{E}_O$ be the corresponding domains. Then*

$$\Gamma(\psi)E_N = E_O.$$

## 4.2 Reflection on $M$

We will characterize the elements $\Theta \in \mathrm{Isom}(M)$ that are a reflection:

**Definition 4.7** (Reflection on $M$)**.** *Given a complete, connected, Riemannian manifold $M$, an isometry $\theta \in \mathrm{Isom}(M)$ is a* ***reflection*** *if there is some $p$ in the fixed-point set $M^\theta$ such that $d\theta_p$ is a hyperplane reflection in the tangent space.*

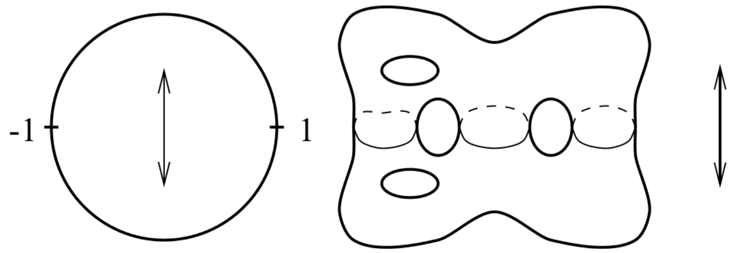
The theory of these reflections is developed in [1] including the following result:

**Proposition 4.8.** *Given a reflection $\theta$ on complete connected Riemannian manifold $M$, we have that the fixed point set $M^\theta$ is a disjoint union of totally geodesic submanifolds, including at least one submanifold of codimension one.*

*Proof.* Let $p \in M^\theta$ be a point such that $T_p\theta$ is a reflection. Then $T_p\theta \circ T_p\theta = \mathrm{id}_{T_pM}$. Thus $\theta$ is an involution. It follows that $T_x\theta$ is a Euclidean involution for each $x \in M^\theta$, and thus it is diagonalizable with eigenvalue 1 on the eigenspace $T_xN$ and $-1$ on the eigenspace $T_xN^\perp$ where $N$ is the connected component of $M^\theta$ containing $x$.

The connected component of $M^\theta$ that contains $p$ has codimension one. □

Any codimension one component of $M^\theta$ is called a **reflection hypersurface**. The image on the top of the next page is from [1] and illustrates examples of reflections generated by multiple hypersurfaces. Note that this only possible for manifolds that are not simply connected. The manifold on the left of the image is $S^1$, not to be confused with the disk. The image is followed by examples of reflections in the sense of Definition 4.7.




**Example 4.9** (Static manifold)**.** *Suppose that $M$ is static in the sense of definition 4.1. Fix a particular hypersurface $\Sigma$ to which the global Killing field $\xi$ is orthogonal. Let $\phi_t$ be the one-parameter group of isometries determined by $\xi$. Define $t : M \to \mathbb{R}$ by setting $t = 0$ on $\Sigma$ and otherwise defining $t(p) = T$ such that $\phi_T(x) = p$ for some $x \in \Sigma$. Define $\theta$ to map $p \in M$ to the unique point on the same $\xi$-trajectory with $t(\theta(p)) = -t(p)$. Note that none of the constructions in this paper will depend on the arbitrary choice of $\Sigma$.*

*A specific example is the case that $M = \mathbb{R}^d$ with coordinates $(t, \vec{x})$, and $\theta$ is reflection in the plane $\Sigma = M^\theta = \{(t, \vec{x}) : t = 0\}$. The plane $\Sigma$ is a reflection hypersurface.*

The above is the physically significant class of examples. The following, though, has the mathematical merit that it works on Riemann surfaces (which do not admit Killing fields):

**Example 4.10** (Schottky double)**.** *Let $S$ denote a compact Riemannian manifold which arises as a Schottky double of a bordered Riemannian manifold $T$ with boundary $\partial T$. The Schottky double is equipped with an antiholomorphic involution $\theta$ from $T$ to its mirror image $\bar{T}$. This involution is a reflection in the sense of Definition 4.7. This reflection is considered in detail in [22]. A specific example is the case that $M$ is the Riemann sphere, $\theta(z) = 1/\bar{z}$, and $\Sigma = M^\theta = \{z : |z| = 1\}$ is the unit circle.*

Maintaining our focus on static manifolds, we adopt the following definition for the remainder of the paper:

**Definition 4.11** (Quantizable manifold)**.** *A complete, connected, Riemannian manifold $M$ is called **quantizable** if it is static and equipped with a reflection in the sense of Example 4.9. Such a manifold is decomposed as*

$$M = \Omega_- \sqcup \Sigma \sqcup \Omega_+$$

*where $\Sigma$ is the $t = 0$ hypersurface. The manifold $M$ is equipped with a reflection $\Theta$ that fixes $\Sigma$ and exchanges $\Omega_+$ and $\Omega_-$.*

## 4.3 Reflection positive inner product on $\mathcal{E}_+ \subset \mathcal{E}$

Let $\mathscr{H}$ be a space of test functions over a quantizable manifold and let $\mathscr{H}'$ be its dual equipped with the measure $d\mu$. We have defined $\mathcal{E} = L^2(\mathscr{H}', d\mu)$.

A transformation $\psi \in \mathrm{Isom}(M)$ acts on $\mathcal{E}$ via $\Gamma(\psi)$. Let $\Theta$ be the reflection of definition 4.11. By abuse of notation, we let $\Theta = \Gamma(\Theta)$ denote the action of the reflection on $\mathcal{E}$. We will use this reflection to define reflection positivity.

**Axiom 4.12** (Reflection positivity)**.** *This is one of the Osterwalder-Schrader axioms, see Axioms 2.2. Define $\mathcal{E}_\pm \subset \mathcal{E}$ to be equal to $\mathcal{E}_{\Omega_\pm}$ in the sense previously defined. We say that $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ is reflection positive when*

$$\langle \Theta A, A \rangle_{\mathcal{E}} \geq 0 \quad \textit{for all } A \in \mathcal{E}_+.$$

*Two equivalent definitions are the following:*

1. *If $\Pi_+ : \mathcal{E} \to \mathcal{E}_+$ is the canonical projection, then*

$$\Pi_+ \Theta \Pi_+ \geq 0,$$

*as an operator on $\mathcal{E}$.*

2. *If $S\{f\}$ is the generating function of the measure $d\mu$, then*

$$0 \leq \sum_{i,j=1}^{n} \overline{c_i} c_j S\{f_i - \Theta f_j\},$$

*for every finite sequence $c_j \in \mathbb{C}$ and $f_j$ supported in $\Omega_+$.*

# 5 The Osterwalder-Schrader construction

In this section and section 6, we assume the existence of the Euclidean space $\mathcal{E}$ defined in section 4.1. We show how to construct $\mathcal{E}$ for a quantizable manifold $M$ in sections 7 and 8. In this section, we give the analytic continuation of the Euclidean structure of $\mathcal{E}$ to Lorentzian structure. This construction is originally due to Osterwalder and Schrader [34, 35], and our treatment most closely follows that of A. Jaffe [16, 23].

We begin with a Hilbert space $\mathcal{E}$ equipped with the inner product $\langle \cdot, \cdot \rangle$ and the action of the isometry group $\mathrm{Isom}(M)$. Recall that there exists a subspace $\mathcal{E}_+$ such that $\langle \Theta u, u \rangle \geq 0$ for $u \in \mathcal{E}_+$. We define a bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ on $\mathcal{E}_+$ by

$$\langle u, v \rangle_{\mathcal{H}} = \langle \Theta u, v \rangle \quad \text{for} \quad u, v \in \mathcal{E}_+. \tag{2}$$

By the self-adjointness of $\Theta$ on $\mathcal{E}$, this is a sesquilinear form:

$$\langle A, B \rangle_{\mathcal{H}} = \int \overline{\Theta A} B d\mu = \int \overline{A} \Theta B d\mu = \overline{\langle B, A \rangle_{\mathcal{H}}}. \tag{3}$$

Let $\mathcal{N}$ denote the kernel of $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

**Definition 5.1** (Quantum Hilbert space)**.** *Let $\mathcal{H}$ denote the completion of $\mathcal{E}_+/\mathcal{N}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Let $\Pi : \mathcal{E}_+ \to \mathcal{H}$ denote the natural quotient map, called the **quantization map**. We have an exact sequence:*

$$0 \to \mathcal{N} \hookrightarrow \mathcal{E}_+ \xrightarrow{\Pi} \mathcal{H} \to 0.$$

The space $\mathcal{H}$ is the Hilbert space of quantum states. To each vector $u \in \mathcal{E}_+$, there corresponds a quantum state $\Pi(u) \equiv \hat{u}$.

## 5.1 Quantization of operators

Assume that $T$ is a densely defined, closable operator on $\mathcal{E}$. We give a condition on $T$ such that it induces a well-defined operator $\hat{T}$ on $\mathcal{H}$.

**Proposition 5.2** (Condition for quantization)**.** *Let $T^+ = \Theta T^* \Theta$. Assume that there exists a domain $\mathcal{D}_+ \subset \mathrm{Dom}(T) \cap \mathrm{Dom}(T^+) \cap \mathcal{E}_+$ such that*

$$T : \mathcal{D}_+ \to \mathcal{E}_+ \quad \text{and} \quad T^+ : \mathcal{D}_+ \to \mathcal{E}_+.$$

*Assume that the projection $\Pi(\mathcal{D}_+)$ is dense in $\mathcal{H}$. Then $T$ has a **quantization** $\hat{T} : \mathcal{H} \to \mathcal{H}$ defined by the commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{E}_+ & \xrightarrow{\Pi} & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow \Theta T^* \Theta & & \downarrow \hat{T} & & \\ 0 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{E}_+ & \xrightarrow{\Pi} & \mathcal{H} & \longrightarrow & 0 \end{array}$$

*The adjoint of $\hat{T}$ is given by*

$$\hat{T}^* = (T^+)\hat{}. \tag{4}$$

*Proof.* Suppose $u \in \mathcal{N} \cap \mathcal{D}_+$. Let $S \subset \mathcal{E}_+$ denote a set of vectors in the domain of $\Theta T^* \Theta$ such that the image of this set under $\Pi$ is dense in $\mathcal{H}$. Then

$$0 = \langle \Pi(\Theta T^* \Theta S), \hat{u} \rangle_{\mathcal{H}} = \langle T^* \Theta S, u \rangle_{\mathcal{E}} = \langle \Theta S, Tu \rangle_{\mathcal{E}} = \langle \Pi(S), \Pi(Tu) \rangle_{\mathcal{H}}.$$

Thus $Tu \in \mathcal{N}$ and hence $T$ is well-defined on $\mathcal{D}_+/(\mathcal{D}_+ \cap \mathcal{N})$. This implies that $\hat{T}$ is well-defined on $\Pi(\mathcal{D}_+)$.

For $\hat{u}, \hat{v} \in \Pi(\mathcal{D}_+)$, let $u, v \in \mathcal{E}_+$ denote representatives in the preimage of the projection. we have that

$$\langle \hat{u}, \hat{T} \hat{v} \rangle_{\mathcal{H}} = \langle \Theta u, Tv \rangle_{\mathcal{E}} = \langle T^* \Theta u, v \rangle_{\mathcal{E}} = \big( (\Theta T^* \Theta)\hat{}\, \hat{u}, \hat{v} \big)_{\mathcal{H}}$$

as desired. The operators $\hat{T}$ and $(\Theta T^* \Theta)\hat{}$ extend uniquely to $\mathcal{H}$ from the dense subset $\Pi(\mathcal{D}_+)$. $\square$

**Proposition 5.3** (Contraction property)**.** *Let $T$ be a bounded operator on $\mathcal{E}$ such that $T$ and $T^+ = \Theta T^* \Theta$ preserve $\mathcal{E}_+$. Then*

$$\left\|\hat{T}\right\|_{\mathcal{H}} \leq \|T\|_{\mathcal{E}} \,. \tag{5}$$

*Proof.* We will prove that for all $\hat{u} \in \mathcal{H}$, $\left\|\hat{T}\hat{u}\right\|_{\mathcal{H}} \leq \|T\|_{\mathcal{E}} \|\hat{u}\|_{\mathcal{H}}$. We use the following lemma:

**Lemma 5.4.** *For all $\hat{u} \in \mathcal{H}$ with preimage $u \in \mathcal{E}$, $\|\hat{u}\|_{\mathcal{H}} \leq \|u\|_{\mathcal{E}}$.*

*Proof.* By the Cauchy-Schwarz inequality for $\mathcal{E}$:

$$\|\hat{u}\|_{\mathcal{H}}^2 = \langle \Theta u, u\rangle_{\mathcal{E}} \leq \|u\|_{\mathcal{E}} \|\Theta u\|_{\mathcal{E}} = \|u\|_{\mathcal{E}}^2 \,.$$

□

Using the Cauchy-Schwarz inequality for $\mathcal{H}$:

$$\left\|\hat{T}\hat{u}\right\|_{\mathcal{H}} = \langle \hat{T}\hat{u}, \hat{T}\hat{u}\rangle_{\mathcal{H}}^{1/2} = \langle \hat{u}, \hat{T}^*\hat{T}\hat{u}\rangle_{\mathcal{H}} \leq \|\hat{u}\|_{\mathcal{H}}^{1/2} \left\|\hat{T}^*\hat{T}\hat{u}\right\|_{\mathcal{H}}^{1/2} .$$

Iterating the application of the Cauchy-Schwarz inequality gives that

$$\begin{aligned}
\left\|\hat{T}\hat{u}\right\|_{\mathcal{H}} &\leq \|\hat{u}\|_{\mathcal{H}}^{1-2^{-n}} \left\|(\hat{T}^*\hat{T})^{2^{n-1}}\hat{u}\right\|_{\mathcal{H}}^{2^{-n}} \\
&= \|\hat{u}\|_{\mathcal{H}}^{1-2^{-n}} \left\|\Pi[(T^+T)^{2^{n-1}}u]\right\|_{\mathcal{H}}^{2^{-n}} \\
&\leq \|\hat{u}\|_{\mathcal{H}}^{1-2^{-n}} \left\|(T^+T)^{2^{n-1}}u\right\|_{\mathcal{E}}^{2^{-n}} \\
&\leq \|\hat{u}\|_{\mathcal{H}}^{1-2^{-n}} \left\|T^+T\right\|_{\mathcal{E}}^{1/2} \|u\|_{\mathcal{E}}^{2^{-n}} \\
&= \|\hat{u}\|_{\mathcal{H}}^{1-2^{-n}} \|T\|_{\mathcal{E}} \|u\|_{\mathcal{E}}^{2^{-n}} \,,
\end{aligned}$$

which gives the desired result. □

## 5.2 Examples of quantizable operators

In this section, we describe two classes of operators that satisfy the quantization condition of Proposition 5.2. The first class are particular unitary operators that quantize to self-adjoint operators:

**Proposition 5.5** (Unitary to self-adjoint)**.** *Let $U$ be a unitary operator on $\mathcal{E}$ that preserves $\mathcal{E}_+$. If $\Theta U^{-1}\Theta = U$, then $U$ admits a quantization $\hat{U}$ and $\hat{U}$ is self-adjoint.*

*Proof.* The operator $\Theta U^*\Theta = U$ preserves $\mathcal{E}_+$, so Proposition implies that $\hat{U}$ is well-defined. Self-adjointness follows from Equation 4: $\hat{T}^* = (T^+)\hat{}$. □

The second class is particular unitary operators that quantize to unitary operators:

**Proposition 5.6** (Unitary to unitary)**.** *Let $U$ be a unitary operator on $\mathcal{E}$ such that $U$ and $U^{-1}$ preserve $\mathcal{E}_+$. If $[U, \Theta] = 0$, then $U$ admits a quantization $\hat{U}$ and $\hat{U}$ is unitary.*

*Proof.* By assumption, the operator $\Theta U^* \Theta = U^{-1}$ preserves $\mathcal{E}_+$. Thus $U$ has a quantization. Similarly, $U^{-1}$ has a quantization. We have that $(U^{-1})\hat{}$ is the inverse of $\hat{U}$, and Equation 4 implies that $\hat{U}^* = \hat{U}^{-1}$. □

The first class of operators, unitary operators $U$ such that $\Theta U^{-1} \Theta = U$, arise from the reflected isometries on $M$.

**Definition 5.7** (Reflected isometry)**.** *An isometry $\psi \in \mathrm{Isom}(M)$ is **reflected** if $\psi^{-1}\Theta = \Theta\psi$. If $\psi$ preserves $\Omega_+$, then $\Gamma(\psi)$ preserves $\mathcal{E}_+$. Then, Proposition 5.5 implies that $\hat{\Gamma}(\psi)$ exists and is self-adjoint. If $\psi$ is reflected then so is $\psi^{-1}$, and $\hat{\Gamma}(\psi^{-1})$ is the inverse of $\Gamma(\psi)$.*

The set of all such isometries is denoted $G_{\mathrm{R}}$. This set is closed under inverses and contains the identity. It is not closed under multiplication.

Unitary operators $U$ such that $[U, \Theta] = 0$ arise from reflection-invariant isometries on $M$:

**Definition 5.8** (Reflection-invariant isometry)**.** *A **reflection-invariant** isometry is an isometry $\psi \in \mathrm{Isom}(M)$ that commutes with $\Theta \in \mathrm{Isom}(M)$. It follows that $[\Gamma(\psi), \Theta] = 0$. If $\psi$ and $\psi^{-1}$ preserve $\Omega_+$, then $\Gamma(\psi)$ and $\Gamma(\psi^{-1})$ preserve $\mathcal{E}_+$. Then, Proposition 5.6 implies that $\hat{\Gamma}(\psi)$ exists and is unitary.*

The set of all such isometries is denoted $G_{\mathrm{RI}}$. It is the stabilizer of a $\mathbb{Z}_2$ action and thus a subgroup of $\mathrm{Isom}(M)$. We have that

$$G_{\mathrm{R}} \cap G_{\mathrm{RI}} = \{\mathrm{id}, \Theta\} \subset G_{\mathrm{R}} \cup G_{\mathrm{RI}} \neq \mathrm{Isom}(M).$$

The reflection-invariant and reflected isometries describes the structure of $\mathrm{Isom}(M)$ in a powerful way:

**Proposition 5.9.** *Let $G^0$ denote the connected component of the identity in $\mathrm{Isom}(M)$. The group $G^0$ is algebraically generated by $G_R \cup G_{RI}$.*

This is a proof on the level of Lie algebras that follows from a Cartan decomposition. Let $\mathfrak{g} = \mathrm{Lie}(\mathrm{Isom}(M))$ be the Lie algebra of Killing fields. The reflection $\Theta$ on $M$ acts on these Killing fields by push forward: $X \mapsto \Theta_* X = (\Theta^{-1})^* X \Theta^*$. This is a Lie algebra homomorphism that squares to the identity. Therefore $\mathfrak{g}$ can be decomposed as a vector space into $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\Theta_*$ is the identity on $\mathfrak{g}_+$ and gives multiplication by $-1$ on $\mathfrak{g}_-$. By the fact that $\Theta_*$ is a Lie algebra homomorphism, we have that

$$[\mathfrak{g}_+, \mathfrak{g}_+] \subset \mathfrak{g}_+, \quad [\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-, \quad [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+.$$

The vector fields in $\mathfrak{g}_+$ are exactly the generators of reflection-invariant isometries, and $\mathfrak{g}_+$ is the Lie algebra of the Lie group $G_{\mathrm{RI}}$. Likewise, vector fields in $\mathfrak{g}_-$ are the generators of the reflected isometries, although $\mathfrak{g}_-$ is not a Lie algebra.

To prove that $G_{\mathrm{R}} \cup G_{\mathrm{RI}}$ algebraically generates $G^0$, recall that $\exp(\mathfrak{g})$ algebraically generates $G^0$. The result follows from the vector space decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$.

## 5.3 Quantization domains

The previous section leaves open the following issue: Suppose that $\psi \in \mathrm{Isom}(M)$ is, for instance, reflected so that $\psi^{-1}\Theta = \Theta\psi$. It is possible that the intersection of $\mathcal{E}_+$ and the preimage of $\mathcal{E}_+$ under $\psi$ is a proper subset $O \subset \mathcal{E}_+$ such that $\Pi(O)$ is not dense in $\mathcal{H}$. This issue of domains is an obstacle to quantizing otherwise well-behaved operators, and it is the major issue that we will confront in section 6.

As an aside, in this section, we introduce the topic of **quantization domains**. A quantization domain is a set $\Omega \subset \Omega_+$ such that $\Pi(\mathcal{E}_\Omega)$ is dense in $\mathcal{H}$. If the set $O$ of the previous paragraph could be proved to correspond to $\mathcal{E}_\Omega$ for some quantization domain $\Omega$, then the issue would be resolved. The problem of classifying quantization domains is believed to be open, and quantization domains are the subject of current research [20]. The following theorem due to A. Jaffe and G. Ritter [23] uses reflected and reflection-invariant operators to find quantization domains:

**Theorem 5.10.** *Let $\psi \in \mathrm{Isom}(M)$ be either reflected or reflection-invariant. Let $\Omega = \psi(\Omega_+)$ and suppose that $\Omega \subset \Omega_+$. Then $\Omega$ is a quantization domain.*

*Proof.* By remark 4.6, we have that $\mathcal{E}_\Omega = \Gamma(\psi)\mathcal{E}_+$.

We will prove that the orthogonal complement in $\mathcal{H}$ of $\Pi(\mathcal{E}_\Omega)$ is zero, which proves that $\Pi(\mathcal{E}_\Omega)$ is dense in $\mathcal{H}$. Let $\hat{u} \in (\Pi(\mathcal{E}_\Omega))^\perp$ with preimage $u \in \mathcal{E}_+$. Let $v \in \mathcal{E}_+$. Then

$$0 = \langle \hat{u}, \Pi(\Gamma(\psi)v) \rangle_{\mathcal{H}} = \langle \Theta u, \Gamma(\psi)v \rangle_{\mathcal{E}}.$$

We have that $\Gamma(\psi)^{-1} = \Gamma(\psi^{-1})$ is unitary, so

$$0 = \langle \Gamma(\psi^{-1})\Theta u, v \rangle_{\mathcal{E}}.$$

First, suppose that $\psi$ is reflection-invariant, i.e. $[\Gamma(\psi^{-1}), \Theta] = 0$. This implies by Proposition 5.6 that $\hat{\Gamma}(\psi)$ is unitary. Furthermore

$$0 = \langle \Gamma(\psi^{-1})\Theta u, v \rangle_{\mathcal{E}} = \langle \Theta\Gamma(\psi^{-1})u, v \rangle_{\mathcal{E}} = \left\langle \hat{\Gamma}(\psi^{-1})\hat{u}, \hat{v} \right\rangle_{\mathcal{H}}.$$

Because this statement holds for all $v \in \mathcal{E}_+$, this implies that $\hat{u} \in \ker \hat{\Gamma}(\psi^{-1})$. Because $\hat{\Gamma}(\psi)$ is unitary, this implies $\hat{u} = 0$, as desired.

Second, suppose that $\psi$ is reflected, i.e. $\Gamma(\psi)\Theta = \Theta\Gamma(\psi^{-1})$. We know from Proposition 5.5 that $\hat{\Gamma}(\psi)$ exists and is self-adjoint on $\mathcal{H}$. Similarly to the above, let $\hat{u} \in \ker \hat{\Gamma}(\psi)$. Now, let $\{\psi_s : s \in \mathbb{R}\}$ be a strongly continuous one-parameter semigroup of isometries such that $\psi = \psi_t$. Then $R(s) = \hat{\Gamma}(\psi_s)$ is a semigroup, self-adjoint, a contraction, and strongly continuous. The last two properties follow from Proposition 5.3. By Stone's theorem (see Appendix A), there exists self-adjoint $K$ such that $\psi_s = e^{-sK}$. Evidently, $\psi_s$ then has zero kernel. □

**Example 5.11** (Positive-time half-space)**.** *In the case that $M = \mathbb{R}^d$ and $T \in \mathbb{R}$, then*

$$\Omega = \{(t, \vec{x}) \in \mathbb{R}^d : t > T\}$$

*is a quantization domain.*

## 5.4 The Hamiltonian

The quantization condition of proposition 5.5 enables a beautiful quantization for the time-translation isometry on a quantizable manifold $M$:

**Theorem 5.12.** *Recall that $M$ is a static spacetime Riemannian manifold. Let $\frac{\partial}{\partial t}$ be the Killing field on $M$ that gives time-translation. Let $\phi_t$ be the corresponding one-parameter group of isometries. For $t \geq 0$, $T(t) = \Gamma(\phi_t)$ has a quantization $R(t)$. This is a strongly continuous one-parameter semigroup of self-adjoint contraction operators on $\mathcal{H}$. The semigroup $R(t)$ leaves invariant the vector $\Omega_0 = \hat{\mathbf{1}}$ where $\mathbf{1}$ is the constant function on $M$. Thus there exists a densely defined, positive, self-adjoint operator $H$ such that*

$$R(t) = \exp(-tH) \quad \text{and} \quad H\Omega_0 = 0.$$

*Proof.* For $t \geq 0$, $T(t)$ is a reflected isometry such that $T(t)\mathcal{E}_+ \subset \mathcal{E}_+$. Thus $R(t) = \hat{T}(t)$ is a self-adjoint semigroup on its domain of definition in $\mathcal{H}$. The contraction property of $R(t)$ follows from Proposition 5.3. The group $T(t)$ is strongly continuous. From this and the contraction property, the strong continuity of $R(t)$ follows. It is evident that $R(t)\Omega_0 = \Omega_0$. The proof then follows from Stone's theorem (see Appendix A). □

This is the Hamiltonian $H$ and ground-state $\Omega_0$ of quantum field theory, as required by the Wightman Axioms (Axioms 2.1). The Hamiltonian gives time-evolution of the physical states in $\mathcal{H}$ and satisfies the physically crucial positivity condition $H \geq 0$.

# 6 Reflection positivity on the level of group representations

We have seen in the previous section that the one-parameter semi-group corresponding to time translation has a representation on the Hilbert space $\mathcal{H}$ of physical states. For quantum field theory, $\mathcal{H}$ must be equipped with a unitary representation of the full symmetry group $G_{\mathrm{lor}}$ of Lorentzian spacetime. Reflection positivity allows the representation of $G = \mathrm{Isom}(M)$ on $\mathcal{E}$ to be analytically continued to a representation of $G_{\mathrm{lor}}$ on $\mathcal{H}$. We give the construction in this section. Recall that any static Riemannian manifold $M$ can be analytically continued to a static Lorentzian manifold $M_{\mathrm{lor}}$. Let $G_{\mathrm{lor}} = \mathrm{Isom}(M_{\mathrm{lor}})$. We will prove that the Osterwalder-Schrader construction extends to give a representation of the identity component $G^0_{\mathrm{lor}}$ of this Lie group on $\mathcal{H}$.

Let $\{\xi_i : 1 \leq i \leq n\}$ be a basis for $\mathfrak{g}$. We will quantize each of the semigroups $\Gamma(\phi_i(\alpha))$ given by exponentiation of the $\xi_i$. This produces $n$ one-parameter families of operators. We will prove that each of these families is a one-parameter unitary group $U_i(\alpha)$ on $\mathcal{H}$, and that these groups give a representation of the action of $G^0_{\mathrm{lor}}$. The crux of this argument is performing the quantization. Reflection-invariant and reflected isometries will still be at the heart of the discussion, but it is not generally possible, for the $\Gamma(\phi_i(\alpha))$ to satisfy the quantization condition given by Proposition 5.2. In particular, it is not generally true that the quantizations have dense domain of definition in $\mathcal{H}$. For this reason, we cannot apply Stone's theorem, as we did in Theorem 5.12, to prove that the $U_i(\alpha)$ are unitary groups. Instead, we use the theory of symmetric local semigroups developed in [11, 30].

## 6.1 Weakened quantization condition

We again consider the quantization of an unbounded linear operator $T$ on $\mathcal{E}$ with partner operator $T^+ = \Theta T^* \Theta$.

The quantization condition of Proposition 5.2 demands that there exists $\mathcal{D}_+ \subset (\mathcal{E}_+ \cap \mathrm{Dom}(T) \cap \mathrm{Dom}(T^+))$ such that $\Pi(\mathcal{D}_+)$ is dense in $\mathcal{H}$ and $\mathcal{D}_+$ is sent into $\mathcal{E}_+$ by $T$ and $T^+$. We can weaken this condition in the following way (due to Jaffe and Ritter):

**Definition 6.1** (Quantization Condition II)**.** *The operator $T$ satisfies*

1. *The domains $\mathrm{Dom}(T)$ and $\mathrm{Dom}(T^*)$ are dense in $\mathcal{E}$.*
2. *There is a set $\mathcal{D}_+ \subset \mathcal{E}_+$ that is in the intersection of the domains of $T$, $T^+$, $T^+T$, and $TT^+$.*
3. *Each of those four operators, $T$, $T^+$, $T^+T$, and $TT^+$ maps $\mathcal{D}_+$ into $\mathcal{E}_+$.*

**Proposition 6.2** (Quantization II)**.** *Given $T$ satisfying Quantization Condition II, we have*

1. *The operators $T|_{\mathcal{D}_+}$ and $T^+|_{\mathcal{D}_+}$ have quantizations with domain $\Pi(\mathcal{D}_+)$.*
2. *For $u, v \in \mathcal{D}_+$, we have $\langle \hat{u}, \hat{T}\hat{v}\rangle_{\mathcal{H}} = \langle (T^+)\hat{}\,\hat{u}, \hat{v}\rangle$.*

*Proof.* Let $f \in \mathcal{D}_+ \cap \mathcal{N}$ where $\mathcal{N}$ is the null space of $\langle \cdot, \cdot \rangle_{\mathcal{E}}$. We will prove that $\Pi(Tf) = \Pi(T^+f) = 0$. This will prove that $T$ and $T^+$ have quantizations with domain $\Pi(\mathcal{D}_+)$. We have that

$$\langle \Pi(Tf), \Pi(Tf) \rangle_{\mathcal{H}} = \langle \Theta Tf, Tf \rangle_{\mathcal{E}} = \langle f, T^* \Theta Tf \rangle_{\mathcal{E}} = \langle \Pi(f), \Pi(T^+Tf) \rangle_{\mathcal{H}}.$$

Where we have used the fact that $T^+T$ maps $\mathcal{D}_+$ to $\mathcal{E}_+$. The Cauchy-Schwarz inequality on $\mathcal{H}$ gives the result. The proof for $T^+$ is identical and uses the fact that $TT^+$ maps $\mathcal{D}_+$ to $\mathcal{E}_+$.

Now suppose $f, g \in \mathcal{D}_+$ Then

$$\langle \hat{f}, \hat{T}\hat{g} \rangle_{\mathcal{H}} = \langle \Theta f, Tg \rangle_{\mathcal{E}} = \langle T^* \Theta f, g \rangle_{\mathcal{E}} = \langle (T^+)\hat{} f, g \rangle_{\mathcal{H}},$$

as desired. □

In the case that $(T^+)\hat{} = \hat{T}$, then $\hat{T}$ is symmetric:

**Definition 6.3** (Symmetric operator)**.** *The operator $\hat{T}$ on $\mathcal{H}$ is **symmetric** when for all $\hat{u}, \hat{v}$ in* $\mathrm{Dom}(\hat{T})$*, we have*

$$\langle \hat{u}, \hat{T}\hat{v} \rangle_{\mathcal{H}} = \langle \hat{T}\hat{u}, \hat{v} \rangle_{\mathcal{H}}.$$

In the case that $\hat{T}$ is symmetric, this weak quantization condition is sufficient to apply the theory of symmetric local semigroups.

## 6.2 Symmetric local semigroups

The theory of symmetric local semigroups was simultaneously developed by A. Klein and L. Landau in [30] and by J. Fröhlich in [11]. Shortly thereafter, these same groups of authors combined their result with Osterwalder and Schrader's reflection positivity to analytically continue a representation of the Euclidean group to a representation of the Poincaré group [12, 31]. In the current paper, we are presenting a generalization of their result. We recount the theory of symmetric local semigroups using the notation of Klein and Landau:

**Definition 6.4.** *A symmetric local semigroup on a Hilbert space $\mathcal{H}$ consists of $(S(\alpha), D_\alpha, T)$. The parameter $T$ is a real number $T > 0$. For each $\alpha \in [0, T]$, $S(\alpha)$ is a symmetric linear operator with domain* $\mathrm{Dom}(S_\alpha) = D_\alpha \subset \mathcal{H}$*. These objects must satisfy the following properties:*

1. *$D_\alpha \supset D_\beta$ if $\alpha \leq \beta$ and $D = \cup_{\alpha \in [0,T]} D_\alpha$ is dense in $\mathcal{H}$.*
2. *$\alpha \to S_\alpha$ is weakly continuous.*
3. *$S_0 = I$ and $S_\beta(D_\alpha) \subset D_{\alpha-\beta}$ for $0 \leq \beta \leq \alpha \leq T$.*
4. *$S_\alpha S_\beta = S_{\alpha+\beta}$ on $D_{\alpha+\beta}$ for $\alpha, \beta, \alpha+\beta \in [0, T]$.*

This is a relaxation of the situation studied by Nussbaum in which the semigroup operators $S_\alpha$ are densely defined and there is a common dense domain on which the semigroup property holds (see [33]).

Then we have the theorem:

**Theorem 6.5.** *For a symmetric local semigroup $(S_\alpha, D_\alpha, T)$, there exists a unique self-adjoint operator $A$ such that*

$$D_\alpha \subset \mathrm{Dom}(e^{-\alpha A}) \quad \textit{and} \quad S_\alpha = e^{-\alpha A}|_{D_\alpha}$$

*for all $\alpha \in [0, T]$.*

**Remark 6.6.** *The authors who prove this result also prove that*

$$\hat{D} \equiv \bigcup_{0<\alpha\leq\gamma} \left[ \bigcup_{0<\beta<\alpha} S_\beta(D_\alpha) \right], \ \textit{where } 0 < S \leq T,$$

*is a core for $A$, i.e. $(A, \hat{D})$ is essentially self-adjoint.*

## 6.3 A unitary representation for $G_{\mathrm{lor}}$

Recall that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ can be decomposed as a vector space into $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ where the Killing fields in $\mathfrak{g}_+$ generate reflection-invariant isometries and the Killing fields in $\mathfrak{g}_-$ generate reflected isometries. We use the theorem of previous section to prove that:

**Theorem 6.7.** *Let $\xi$ be a Killing filed in $\mathfrak{g}_+$ or $\mathfrak{g}_-$ that generates the one-parameter isometry group $\{\phi_\alpha\}$. There exists a densely defined, self-adjoint operator $A_\xi$ on $\mathcal{H}$ such that*

$$\hat{\Gamma}(\phi_\alpha) = \begin{cases} e^{-\alpha A_\xi} & \textit{if } \xi \in \mathfrak{g}_-, \\ e^{i\alpha A_\xi} & \textit{if } \xi \in \mathfrak{g}_+. \end{cases}$$

*Proof.* For the first case, let $\xi \in \mathfrak{g}_-$. Then each isometry $\phi_\alpha$ is reflected. Define $\Omega_\alpha \equiv \phi_\alpha^{-1}(\Omega_+)$. We have that $\phi_0$ is the identity map, so $\Omega_0 = \Omega_+$. The continuity of $\phi_\alpha(x)$ with respect to $\alpha$ implies that for $\alpha$ in some neighborhood of zero, $\Omega_\alpha$ is a nonempty open subset of $\Omega_+$. As $\alpha \to 0$ from above, $\Omega_\alpha$ increases to fill $\Omega_+$.

By remark 4.6, we have that $\Gamma(\phi_\alpha)\mathcal{E}_{\Omega_\alpha} \subset \mathcal{E}_+$. Then, by Proposition 6.2, $\Gamma(\phi_\alpha)$ quantizes to an operator $\hat{\Gamma}(\phi_\alpha)$ with domain $D_\alpha \equiv \Pi(\mathcal{E}_{\Omega_\alpha})$. By the fact that $\phi_\alpha$ is reflected, $\hat{\Gamma}(\phi_\alpha)$ is symmetric.

Fix $\gamma > 0$. Then

$$\bigcup_{0<\alpha<\gamma} \Omega_\alpha = \Omega_+ \Rightarrow \bigcup_{0<\alpha<\gamma} \mathcal{E}_{\Omega_\alpha} = \mathcal{E}_+.$$

It follows that $D = \cup_{0<\alpha<\gamma} D_\alpha$ is dense in $\mathcal{H}$. It is routine to verify that $(\hat{\Gamma}(\phi_\alpha), D_\alpha, \gamma)$ satisfies Definition 6.4 for a symmetric local semigroup. Theorem 6.5 then gives the desired result.

For the second case, let $\xi \in \mathfrak{g}_+$. Then each isometry $\phi_\alpha$ is reflection-invariant, and we have that $\Gamma(\phi_\alpha)^+ = \Gamma(\phi_{-\alpha})$ on $\mathcal{E}$. We claim that $\Gamma(\phi_\alpha)\mathcal{E}_+ \subset \mathcal{E}_+$. Suppose that this is true. Then, $\Gamma(\phi_\alpha)$ satisfies Quantization Condition I, given by Proposition 5.2. Then $\hat{\Gamma}(\phi_\alpha)$ is defined on $\Pi(\mathcal{E}_+)$ which is dense in $\mathcal{H}$ be definition. We have that $\hat{\Gamma}(\phi_\alpha)$ extends by continuity to a one-parameter unitary group on $\mathcal{H}$. By Stone's theorem (see Appendix A), there exists a self-adjoint operator $A$ on $\mathcal{H}$ such that

$$\hat{\Gamma}(\phi_\alpha) = \exp(i\alpha A).$$

To complete the proof in this case, we simply need to prove that $\Gamma(\phi_\alpha)\mathcal{E}_+ \subset \mathcal{E}_+$. We will prove that any reflection-invariant isometry $\psi$ preserves $\Sigma$ and either preserves or exchanges $\Omega_+$ and $\Omega_-$. The claim then follows from the fact that $\phi_\alpha$ is in the identity component of $G$.

Suppose that $p \in \Sigma$ and, WLOG, $\psi(p) \in \Omega_+$. Then, by reflection-invariance, $\Omega_+$ contains $(\Theta\psi\Theta)(p) = (\Theta\psi)(p) \in \Omega_-$, which is a contradiction. Thus $M$ restricts to an isometry on $\Sigma$ and an isometry on $\Omega_+ \sqcup \Omega_-$. This gives the desired result. □

Now let $\{\xi_i^+ : 1 \leq i \leq n_+\}$ be a basis of $\mathfrak{g}_+$ and let $\{\xi_i^- : 1 \leq i \leq n_-\}$ be a basis of $\mathfrak{g}_-$. Let $A_i^\pm$ denote the corresponding densely defined self-adjoint operators on $\mathcal{H}$ constructed in the previous theorem. We define $G'_{\text{lor}}$ as the group generated by the one-parameter unitary groups

$$U_i^\pm(\alpha) = \exp(i\alpha A_i^\pm),$$

and we claim that it is isomorphic to the identity component of $G_{\text{lor}} = \text{Isom}(M_{\text{lor}})$. Because this claim only concerns the identity component, it can be checked on the level of Lie algebras.

If $\xi \in \mathfrak{g}$ generates $\psi_\alpha$, then let $\Gamma(\xi)$ denote the generator of $\Gamma(\psi_\alpha)$. We have that $\Gamma([X,Y]) = [\Gamma(X), \Gamma(Y)]$, so we have that $\Gamma$ gives a Lie group homomorphism. Then let $\hat{\Gamma}(\xi)$ denote the generator of $\hat{\Gamma}(\psi_\alpha)$. By the above analysis, we have that $\hat{\Gamma}(\xi_j^+) = iA_j^+$ and $\hat{\Gamma}(\xi_j^-) = -A_j^-$. Then define the two Lie algebras:

$$\hat{\mathfrak{g}}_\pm \equiv \{\hat{\Gamma}(X) : X \in \mathfrak{g}_\pm\}.$$

We have that $\hat{\mathfrak{g}}_+ \oplus i\hat{\mathfrak{g}}_-$ is a Lie algebra represented by skew-symmetric operators on $\mathcal{H}$. Then our claim is:

**Theorem 6.8.** *Let $\mathfrak{g}_{\text{lor}}$ be the Lie algebra of the connected component of* $\text{Isom}(M_{\text{lor}})$. *There is an isomorphim of Lie algebras*

$$\mathfrak{g}_{\text{lor}} \cong \hat{\mathfrak{g}}_+ \oplus i\hat{\mathfrak{g}}_-.$$

*Proof.* Given coordinates $x^\mu$ on $M$, let $M_\mathbb{C}$ denote the manifold obtained by allowing the $x^0$ coordinate to take values in $\mathbb{C}$. This manifold is equipped with the involution $J : M_\mathbb{C} \to M_\mathbb{C}$ given by $x^0 \mapsto -ix^0$. Denote the induced map on Lie algebras by $J^*$. The Lie algebra $\mathfrak{g}_{\text{lor}}$ is generated by

$$\{\xi_j^+\}_{1\le j\le n_+} \cup \{\eta_k\}_{1\le k\le n_-}, \quad \text{where} \quad \eta_j \equiv iJ^*(\xi_j^-).$$

We compute the commutation relations of this algebra. Let $f_{ijk}$ be the set of real structure constants such that

$$[\xi_i^-, \xi_j^-] = \sum_{k=1}^{n_+} f_{ijk}\xi_k^+.$$

Applying $J^*$ to both sides gives

$$[\eta_i, \eta_j] = -\sum_{k=1}^{n_+} f_{ijk}\xi_k^+.$$

These, together with the inherited relations for $\mathfrak{g}_+$ are precisely the commutation relations of $\hat{\mathfrak{g}}_+ \oplus i\hat{\mathfrak{g}}_-$, which proves the result. □

# 7 Construction of reflection positive measures

The constructions of the previous sections are the achievement of reflection positivity. The remainder of this exposition will focus on constructing the reflection positive forms that enable these arguments to succeed. In this section, we will construct a family of reflection positive measures $d\mu$ that give reflection positive forms on $\mathcal{E} \equiv L^2(\mathscr{H}', d\mu)$.

The situation is that we begin with the Hilbert space $L^2(M)$ and introduce test functions in order to give it the structure of a **rigged Hilbert space**:

$$\mathscr{H}(M) \subset L^2(M) \subset \mathscr{H}'(M), \tag{6}$$

where $\mathscr{H}(M)$ is a **nuclear space** and $\mathscr{H}'(M)$ is its continuous dual. These ideas will be defined in section 7.1. In section 7.2, we construct this rigging for a wide class of quantizable manifolds $M$. This construction replaces the incorrect construction in [23, 24].

The nuclear structure of $\mathscr{H}(M)$ allows us to construct the measure $d\mu$ on $\mathscr{H}'$. This measure must satisfy the Euclidean invariance condition of axiom 4.4 and the reflection positivity condition of axiom 4.12. As a sequence of propositions over the course of sections 7.4 and 7.5, we prove the following theorem:

**Theorem 7.1.** *Let $C$ be a positive, continuous, nondegenerate bilinear form on the nuclear function space $\mathscr{H}(M)$. Then there exists a unique Gaussian measure $d\mu_C$ on $\mathscr{H}'(M)$ with covariance $C$ and mean zero. This Gaussian* *measure is Euclidean invariant if and only if $C$ is Euclidean invariant in the sense that for $\psi \in \operatorname{Isom}(M)$ and $f \in \mathscr{H}$,*

$$\langle \psi_* f, C\psi_* f\rangle_{L^2(M)} = \langle f, Cf\rangle_{L^2(M)}.$$

*This Gaussian measure satisfies reflection positivity if and only if $C$ is reflection positive in the sense that*

$$\langle \Theta f, Cf\rangle_{L^2(M)} \geq 0,$$

*for functions $f \in \mathscr{H}$ supported at positive times.*

This pushes the issue of reflection positive to bilinear forms on $L^2(M)$. An example of a Euclidean invariant, reflection positive form $C$ is constructed in section 8.

## 7.1 Nuclear spaces

We will say what it means for a topological vector space to be nuclear. Several slightly different definitions abound in the literature. Our definition follows [13, Chapter 1].

**Definition 7.2.** *Let $\mathscr{H}$ be a topological vector space equipped with a countable family of inner products denoted $\langle\cdot,\cdot\rangle_n$ for $n = 1, 2, \ldots$. Suppose that the inner products give the topology of $\mathscr{H}$ in the sense that a neighborhood basis of zero is given by the sets $U_{n,\epsilon} = \{f \in \mathscr{H} : \|f\|_n < \epsilon\}$. Let $H_n$ denote the Hilbert space given by the completion of $\mathscr{H}$ under the $n^{th}$ inner product. The space $\mathscr{H}$ is complete relative to the aforementioned topology if and only if it can be written*

$$\mathscr{H} = \bigcap_n H_n.$$

*In this case, $\mathscr{H}$ is called a **countably Hilbert space**. Suppose furthermore that*

$$\|f\|_n \leq \|f\|_{n+1} \qquad \text{for all } f \in \mathscr{H}.$$

*If this condition does not hold, then the inner products can be redefined as*

$$\langle f, g\rangle'_n \equiv \sum_{i=1}^{n} \langle f, g\rangle_i.$$

*Then the embedding $H_n \hookrightarrow H_{n+1}$ is a continuous map from an everywhere dense set to an everywhere dense set. Extend it to a continuous linear map $T^n_{n+1} : H_{n+1} \to H_n$. We define $T^n_m$ to be the map achieved in this way for*

*$m > n$. It is well-defined. The space $\mathscr{H}$ is **nuclear** if the following condition holds: for all $m$, there exists $n > m$ such that $T_m^n$ can be written*

$$T_m^n f = \sum_{i=1}^{\infty} \lambda_k \langle f, u_k \rangle v_k \quad \text{for all } f \in H_m,$$

*where $\{u_k\}$ and $\{v_k\}$ are orthonormal systems of vectors in $H_m$ and $H_n$ respectively, $\lambda_k > 0$, and $\sum_k \lambda_k < \infty$. This is the condition that $T_m^n$ is a **nuclear operator**. In the case that $\mathscr{H}$ is a Hilbert space, we instead say that $T_m^n$ is a **trace class operator**.*

**Remark 7.3.** *For a Hilbert space $\mathscr{H}$, some authors substitute the condition that for all $m$, there exists $n$ such that $T_m^n$ is a **Hilbert-Schmidt operator**. This is essentially the same definition, because every trace class operator is Hilbert-Schmidt, and the composition of two Hilbert-Schmidt operators is trace class.*

**Proposition 7.4.** *Give a countably Hilbert space and, in particular, a nuclear space*

$$\mathscr{H} = \bigcap_n H_n,$$

*then its continuous dual $\mathscr{H}'$ is*

$$\bigcup_n H_n' \cong \bigcup_n H_n.$$

*Proof.* The isomorphism comes from the fact that each $H_i$ is a Hilbert space. We prove that $\mathscr{H}' = \bigcup H_n'$. Let $\phi \in H_n$, then by construction $\phi$ is continuous with respect to the topology of $H$, so $\phi \in \mathscr{H}'$. Given $\psi \in \mathscr{H}'$, the fact that $\psi$ is continuous on $\mathscr{H}$ implies that there exists $\epsilon$ and $n$ such that $\psi$ is bounded in the sphere $\|f\|_n \leq \epsilon$. Then $\psi$ is continuous with respect to the norm $\|\cdot\|_n$. □

## 7.2 Construction of nuclear space over $M$

Let $M$ be a quantizable manifold in the sense of definition 4.11, and suppose that it satisfies the additional constraint that the spatial hypersurface $\Sigma$ is compact. Then, we can construct the following nuclear space of test functions over $M$. Note that for transparency of the construction, the indexing differs from the above definition, but it is equivalent.

**Proposition 7.5.** *Let $\Delta_M$ be the Laplace-Beltrami operator on $L^2(M)$. Let $Q_t$ denote the operator on $L^2(M)$ that gives multiplication by the global time coordinate. Then let*

$$H = \frac{1}{2}\left(-\Delta_M + Q_t^2\right) + \frac{1}{2}.$$

*This is a hybrid between the Laplacian and the Harmonic oscillator operator. For integer $n$, define the following inner product on $L^2(M)$:*

$$\langle f, g\rangle_n = \langle f, (H+I)^n g\rangle.$$

*Let $H_n$ be the Hilbert space achieved by completion under $\langle\cdot,\cdot\rangle_n$. Then*

$$\bigcap_{n=-\infty}^{\infty} H_n$$

*is a nuclear space.*

*Proof.* The Laplace-Beltrami operator on $M$ can be written as

$$\Delta = \Delta_\Sigma + \partial_t^2,$$

where $\Delta_\Sigma$ is the Laplace-Beltrami operator on $\Sigma$.

The compactness of $\Sigma$ gives us that the positive operator $-\Delta_\Sigma$ has a countable set of eigenfunctions $g_j$ that give an orthonormal basis for the Hilbert space $L^2(\Sigma)$. Let $\lambda_j$ be the corresponding eigenvalues with multiplicity. Arrange the indices such that $\lambda_j \leq \lambda_{j+1}$ for all $j$.

The single-dimensional Harmonic oscillator operator $\frac{1}{2}(-\partial_t^2 + t^2) - \frac{1}{2}$ has the normalized Hermite polynomials as eigenfunctions. Denoted $h_0, h_1, \ldots$, these Hermite functions are an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$ with corresponding eigenvalues $0, 1, \ldots$.

Note that $L^2(M) = L^2(\Sigma)\hat{\otimes}L^2(\mathbb{R})$ where $\hat{\otimes}$ denotes the completed Tensor product. Then the operator $H$ can be written as

$$H = -\frac{1}{2}\Delta_\Sigma \otimes \mathrm{id} + \mathrm{id} \otimes \frac{1}{2}(-\partial_t^2 + t^2 + 1).$$

By the previous discussion, $H$ has eigenvalues $g_j \otimes h_k$, and they give an orthonormal basis for the Hilbert space $L^2(M)$.

We claim that for any $\varepsilon > 0$, $(H+I)^{-[(d+1)/2+\varepsilon]}$ is trace class. We use Weyl's asymptotic formula [5, Page 172], which implies that

$$\lambda_k \sim \mathrm{const} \cdot k^{2/(\dim\Sigma)} = \mathrm{const} \cdot k^{2/(d-1)},$$

and we use the inequality

$$\sum_{j=0}^{\infty} (a+j)^{-b} \leq \mathrm{const} \cdot a^{-(b-1)},$$

valid for $1 < a$, $1 < b$ to successively bound the asymptotic behavior of the two sums

$$\mathrm{tr}\left[(H+I)^{-[(d+1)/2+\varepsilon]}\right] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (\lambda_j + k + 1)^{-[(d+1)/2+\varepsilon]}.$$

(Note that $d = 2$ is a special case in which we only apply the inequality once.)

For $n \in \mathbb{Z}$, define the inner product $\langle \phi, \psi \rangle_n = \langle \phi, (H+I)^n \psi \rangle_{L^2(M)}$. Let $\mathscr{H}_n$ be the Hilbert space that comes from completion of the $n^{th}$ inner product. Because $0 \leq H$, $\|\phi\|_n \leq \|\phi\|_{n+1}$, so there is an injection $i^n_{n+1} : \mathscr{H}_{n+1} \hookrightarrow \mathscr{H}_n$.

Note that $(H+I)^{1/2} \circ i^n_{n+1} : \mathscr{H}_{n+1} \hookrightarrow \mathscr{H}_n$ is a unitary map. Then,

**Lemma 7.6.** *For $D > (d+1)$, the canonical injection*

$$i^n_{n+D} = i^n_{n+1} \circ \cdots \circ i^{n+D-1}_{n+D} : \mathscr{H}_{n+D} \hookrightarrow \mathscr{H}_n$$

*is trace class.*

*Proof.* Write

$$i^n_{n+D} = (H+I)^{-D/2} \left[ (H+I)^{D/2} i^n_{n+D} \right].$$

The second term is unitary and the first term is trace class. This gives the result. □

The proposition then follows follows from our previous definition of a nuclear space. □

In the case that $M = \mathbb{R}^d$, then it is possible to instead use $\mathscr{H} = \mathcal{S}(\mathbb{R}^d)$. This space has a very similar nuclear structure to the structure defined above:

**Remark 7.7.** *Define the operator $H$ on $L^2(\mathbb{R}^d)$ as*

$$H = \sum_{k=1}^{d} \frac{1}{2} \left[ -\partial_k^2 + Q_k^2 \right] - \frac{1}{2},$$

*where the operator $Q_k$ is given by multiplication by $x_k$. The operator $H$ is diagonalized in the basis of Hermite functions. For $f$ and $g$ finite linear combinations of Hermite functions, let*

$$\langle f, g \rangle_n = \langle f, (H+I)^n g \rangle.$$

*This inner product extends by completion to define a Hilbert space $\mathscr{H}_n$. Because $0 \leq H$, we have $\|f\|_n \leq \|f\|_{n+1}$, so there is an inclusion $i^n_{n+1} : \mathscr{H}_{n+1} \hookrightarrow \mathscr{H}_n$. By a similar analysis to the previous proposition, we have that for $D > 2d$, the canonical injection*

$$i^n_{n+D} = i^n_{n+1} \circ \cdots \circ i^{n+D-1}_{n+D} : \mathscr{H}_{n+D} \hookrightarrow \mathscr{H}_n$$

*is trace class. Schwartz space is given by $\bigcap_n \mathscr{H}_n$.*

## 7.3 Gaussian measures

Now we define the measure $d\mu$ on the space $\mathscr{H}'$. There is no $\sigma$-finite Lebesgue measure on an infinite dimensional space, so we will use infinite-dimensional Gaussian measures. This is a concept that we will define.

For finite-dimensional spaces, Gaussian measures are defined from the Lebesgue measure in the following way:

**Definition 7.8.** *Let $V$ be a finite-dimensional inner product space. Let $B_0(V)$ be the completion of the Borel $\sigma$-algebra on $V$. Let $\lambda$ be the usual Lebesgue measure on $V$. Then for $A \in B_0(V)$, the* ***standard Gaussian measure*** *is defined by*

$$\gamma(A) = \frac{1}{\sqrt{2\pi}^n} \int_A \exp\left(-\frac{1}{2}\|x\|^2\right) d\lambda(x).$$

**Remark 7.9.** *The standard Gaussian measure on a finite dimensional inner product space is equivalent to its Lebesgue measure.*

To define a Gaussian measure in an infinite-dimensional setting, suppose that $\mathscr{H}$ is a nuclear space and $\mathscr{H}'$ is its dual.

**Definition 7.10.** *For $A$ a Borel set on $\mathbb{R}^n$ and $f_1, \ldots, f_n \in \mathscr{H}$, then*

$$S_{A;f_1,\ldots,f_n} \equiv \{\theta \in \mathscr{H}' : (f_1(\theta), \ldots, f_n(\theta)) \in A\}$$

*is called a* ***Borel cylinder set*** *in $\mathscr{H}'$. The smallest $\sigma$-algebra in $\mathscr{H}'$ that contains all of the cylinder sets is called the* ***cylinder $\sigma$-algebra****.*

This definition can be interpreted in the following way: For $\Psi$ a finite-dimensional subspace of $\mathscr{H}$ containing $f_1, \ldots, f_n$, let $\Psi^0 \subset \mathscr{H}'$ be the linear space of $\theta$ such that

$$f_1(\theta), \ldots, f_n(\theta) = 0.$$

Then the map $\theta \mapsto f_1(\theta), \ldots, f_n(\theta)$ factors through $\mathscr{H}'/\Psi_0$. Choose $A \subset \mathscr{H}'/\Psi_0$ such that the image of $A$ is a Borel subset of $\mathbb{R}^n$. The preimage of $A$ under the canonical projection is a **Borel cylinder set** in $\mathscr{H}'$. We say that it is **based in** $\Psi$. Given measures $d\nu_\Psi$ on $\mathscr{H}'/\Psi_0$ parameterized by all choices of $\Psi$, then these measures are said to be **compatible** if we have the following: for $\Psi_1 \subset \Psi_2$ and $X \subset \mathscr{H}'/\Psi_1^0$ a Borel set,

$$d\nu_{\Psi_1}(X) = d\nu_{\Psi_2}(Q^{-1}(X)),$$

where $Q$ denotes the natural map from $\mathscr{H}'/\Psi_2^0$ into $\mathscr{H}'/\Psi_1^0$. In this case, the measure on the sets $\mathscr{H}'/\Psi_0$ lifts to a measure $d\nu$ on Borel cylinder sets. If each $d\nu_\Psi$ is a Gaussian measure, then $d\nu$ is said to be a **Gaussian measure on cylinder sets** for $\mathscr{H}'$. If $d\nu$ is countably additive on the $\sigma$-algebra of Borel cylinder sets, then it is known that it can be uniquely extended to a countably additive measure $d\mu$ on the class of all Borel sets (see, for instance, [13, IV.2]). This is called a **Gaussian measure for $\mathscr{H}'$**.

## 7.4 Construction of Gaussian measure

Given the groundwork of the previous section we prove the following proposition, which is part of Theorem 7.1. Our proof follows [13] and [16].

**Proposition 7.11.** *Let $C$ be a covariance operator defined on $\mathscr{H}$. Then there is a unique Gaussian measure, which we denote $d\mu_C$, defined on $\mathscr{H}'$, and having $C$ as its covariance operator.*

*Proof.* Let $C_V$ be the restriction of $C$ to an $n$-dimensional subspace $V \subset \mathscr{H}$. Then $C_V$ is the covariance of a unique finite dimensional Gaussian measure on $V$:

$$dx_{C_V} = \frac{\det C_V^{-1/2}}{\sqrt{2\pi}^n} \exp\left(-\frac{1}{2}\langle x, C_V^{-1}x\rangle_0\right) dx \tag{7}$$

where $dx$ is the Lebesgue measure on $V$. Note that we use the nuclear inner product $\langle\cdot,\cdot\rangle_0$ on $H_0$, which is equal in our case to the inner product on $L^2(M)$.

Let $V'$ be the dual space of $V$ under the $H_0$ inner product. The inner product relates these spaces and allows us to think of $dx_{C_V}$ as a measure on $V'$. As above, define $V^0 = \{\theta \in \mathscr{H}' : \langle\theta, V\rangle_{H_0} = 0\}$. Then the Gram-Schmidt process gives rise to a unique $H_0$-orthogonal decomposition $\mathscr{H}' = V' \oplus V^0$, i.e. $V'$ is isomorphic to $\mathscr{H}'/V^0$. Thus $dx_{C_V}$ gives a Gaussian measure on $\mathscr{H}'/V^0$. The measures achieved in this way for different subspaces $V$ are compatible in the sense defined above. This follows immediately from the fact that a finite-dimensional Gaussian measure is uniquely defined by its covariance.

Thus we have a measure $d\nu_C$ on the cylinder sets of $\mathscr{H}'$. It is finitely additive and regular by construction. It remains to prove that it is countably additive, in which case it extends to the Borel algebra as discussed above.

Recall the nuclear space structure

$$\mathscr{H} = \bigcap_{n=-\infty}^{\infty} H_n \quad \text{and} \quad \mathscr{H}' = \bigcup_{n=-\infty}^{\infty} H_n,$$

where $H_0 = L^2$. Let $S(r,j) = \{\theta : \|\theta\|_j \le r\}$ be the sphere of radius $r$ in $H_j$. Given our measure $d\nu$ on cylinder sets in $\mathscr{H}'$, we say that it has **vanishing measure at infinity** in $H_j$ if for all $\varepsilon > 0$ there exists $r$ such that for any Borel cylinder set $X$ disjoint from $S(r,j)$, we have that $\nu(X) \le \varepsilon$.

**Lemma 7.12.** *Let $d\nu$ be a finitely additive, regular measure defined on Borel cylinder sets in $\mathscr{H}'$. Suppose that there exists $j$ such that $d\nu$ has a vanishing measure at infinity in $H_j$. Then $d\nu$ defines a countably additive measure on the Borel cylinder sets of $\mathscr{H}'$.*

*Proof.* Let $Y = \bigsqcup_{k=1}^{\infty} Y_k$ be a disjoint union of Borel cylinder sets. Let $Y_0 = \mathscr{H}' \setminus Y$. We wish to prove that $\sum_{k=0}^{\infty} \nu(Y_k) = 1$. By finite additivity, the sum is less than or equal to 1. By regularity, it is sufficient to prove $\sum_{k=0}^{\infty} \nu(Z_k) \geq 1$ where each $Z_k$ is a weakly open cylinder set containing $Y_k$.

Fix $\varepsilon > 0$. The ball $S(r, j)$ is weakly compact, so there is a finite union $Z$ of $Z_k$ such that $Z$ contains $S(r, j)$. By hypothesis, $\varepsilon \geq \nu(\mathscr{H}' \setminus Z) \geq 1 - \sum_{k=0}^{\infty} \nu(Z_k)$, which proves the result. □

If the hypotheses of this lemma hold, then the we have completed the proof. By the continuity of $C$ on $\mathscr{H}$, there exists $j$ such that $C : H_j \to H_{-j}$ boundedly or, equivalently,

$$|\langle f, Cg \rangle_0| \leq \text{const} \, \|f\|_j \, \|g\|_j \, . \tag{8}$$

Fix this $j$ (It is almost our choice of $j$ to satisfy the above lemma. We will increase it slightly over the course of the proof.).

Let $Z$ be a cylinder set based on the finite-dimensional subspace $V \subset \mathscr{H}$ and suppose that $Z \cap S(r, -j) = \emptyset$. Let $S_V = S(r, -j) + V^0$. Because $Z = Z + V_0$, we have that $Z \cap S_V = \emptyset$. This gives us the first of the following inequalities, and we will prove the second:

$$\nu_C(Z) \leq \int_{\mathscr{H}' \setminus S_V} d\nu_C \leq \varepsilon.$$

Recall that $\mathscr{H}' = V' \oplus V^0$. Let $P_V$ denote the canonical projection onto $V'$. Then

$$\int_{\mathscr{H}' \setminus S_V} d\nu_C = (2\pi)^{-\dim V/2} \det C_V^{-1/2} \int_{V' \setminus P_V S(r,-j)} \exp\left(-\frac{1}{2} \langle x, C_V^{-1} x \rangle\right) dx.$$

We simplify the integration through the change of variable

$$Y = V' \setminus C_V^{-1/2} P_V S(r, -j).$$

Then the integral becomes

$$\int_{\mathscr{H}' \setminus S_V} d\nu_C = (2\pi)^{-\dim V/2} \int_Y \exp\left(-\frac{\|y\|^2}{2}\right) dy.$$

By definition, we have that for $y \in Y$,

$$C_V^{1/2} y \in V' \setminus P_V S(r, -j) \subset \mathscr{H}' \setminus P_V S(r, -j) \subset \mathscr{H}' \setminus S(r, -j).$$

Furthermore, we have by the nuclear space structure of $\mathscr{H}$, that $S(r, -j) = (H + I)^{j/2} S(r, 0)$. Thus we have that

$$(H + I)^{-j/2} C_V^{1/2} y \in \mathscr{H}' \setminus S(r, 0).$$

Thus

$$r^2 \leq \langle y, C_V^{1/2} (H + I)^{-j} C_V^{1/2} y \rangle_{H_0}$$

which implies that

$$\nu_C(Z) \leq (2\pi)^{-\dim V/2} \int \langle y, C_V^{1/2}(H+I)^{-j} C_V^{1/2} y\rangle_{H_0} r^{-2} \exp\left(-\frac{\|y\|^2}{2}\right) dy.$$

Let $A_V = C_V^{1/2}(H+I)^{-j}C_V^{1/2}$. The above equation gives that $\nu_C(Z) \leq r^{-2}\operatorname{tr}_V A_V$ where this is the $H_0$ trace taken over $V$. If we can prove that $\operatorname{tr}_V A_V$ is bounded independent of $V$, then by taking $r$ large enough we achieve the desired result. By equation 8, there is a constant $\alpha$, independent of $V$, such that

$$|\langle f, C_V g\rangle_0| \leq \alpha \left\|(H+I)^{j/2} f\right\|_0 \left\|(H+I)^{j/2} g\right\|_0$$

for $f, g \in \mathscr{H}$. Thus we have that for $f, g \in (H+I)^{1/2}\mathscr{H}$,

$$\left|\langle f, (H+I)^{-j/2} C_V (H+I)^{-j/2} g\rangle_0\right| \leq \alpha \|f\|_0 \|g\|_0 .$$

This inequality extends to $f, g \in H_0$ and so $C_V^{1/2}(H+I)^{-j/2}$ is a bounded operator on $H_0$ with norm independent of $V$. By the nuclear property of the $H_j$ norms, we increase $j$ so that $C_V^{1/2}(H+I)^{-j/2}$ is Hilbert-Schmidt, with norm independent of $V$. The required bound on $\operatorname{tr}_V A_V$ follows. This $j$ satisfies Lemma 7.12, so the proof that $d\mu_C$ exists is complete.

The uniqueness of $d\mu_C$ follows from the details of the construction. The uniqueness of finite-dimensional Gaussian measures implies that $d\mu_C$ is unique on cylinder sets. The fact that these generate all Borel sets under repeated monotone limits then gives the result. □

**Corollary 7.13.** *The generating function $S\{f\} = \int e^{i\langle \Phi, f\rangle} d\mu_C$ defined by the Gaussian measure $d\mu_C$ is given by*

$$S\{f\} = \exp\left(-\frac{1}{2}\langle f, Cf\rangle_{H_0}\right). \tag{9}$$

*Proof.* Let $\theta_f \in \mathscr{H}'$ be the dual vector to $f \in \mathscr{H}$. By the previous construction, the generating function is given by a finite Gaussian integral over the one-dimensional space $V$ spanned by $\theta_f$. Suppose that $C|_V\theta = \lambda V$. Then equation 7 for the finite Gaussian integral becomes:

$$(2\pi\lambda)^{-1/2} \int \exp\left(-\frac{1}{2}\langle x, \lambda^{-1}\rangle_{H_0}\right) \exp(ipx) dx = \exp\left(-\frac{1}{2}\langle p, \lambda p\rangle\right)$$

where the integral is evaluated term-by-term in the power series. □

## 7.5 OS axioms for the Gaussian measure

To complete the proof of Theorem 7.1, we show that the Euclidean invariance and reflection positivity of $d\mu$ are equivalent to the Euclidean invariance and reflection positivity of $C$. These conclusions follow from Corollary 7.13, which gives the generating function of the Gaussian measure $d\mu$ as

$$S\{f\} = \exp\left(-\frac{1}{2}\langle f, Cf\rangle_{L^2}\right).$$

Recall from axiom 4.4 that $d\mu$ is Euclidean invariant when for all $\psi \in \mathrm{Isom}(M)$ and $f \in \mathscr{H}$, we have $S\{\psi_* f\} = S\{f\}$. It follows that this is equivalent to the condition

$$\langle \psi_* f, C\psi_* f\rangle_{L^2} = \langle f, Cf\rangle_{L^2}.$$

Recall from axiom 4.12 that $d\mu$ is reflection positive if for any finite collection of functions $f_i$ with positive support in $\mathscr{H}$, we have that

$$M_{ij} = S\{f_i - \Theta f_j\} \tag{10}$$

is positive. By the formula for $S$, this is $S\{f_i\}S\{f_j\}\exp\langle \Theta f_i, Cf_j\rangle$, so if $C$ is reflection positive then $d\mu$ is reflection positive. The converse is valid for non-Gaussian measures and is stated as a separate proposition due to [16]:

**Proposition 7.14.** *Let $d\mu$ be a measure on $\mathscr{H}'$ with generating functional $S\{f\}$. Assume that $S\{f\}$ is an entire analytic function of the complex variable $f \in \mathscr{H}$. If $d\mu$ is reflection positive in the sense of axiom 4.12, then so is the two-point function of $d\mu$.*

*Proof.* Define $M_{ij}$ as in equation 10. Take any real $f \in \mathscr{H}$. Let $f_1 = \lambda f$, $f_2 = 0$, $\alpha_1 = \lambda^{-1}$, and $\alpha_2 = -\lambda^{-1}$. Then

$$0 \le \sum_{i,j=1}^{2} \alpha_i M_{ij} \alpha_j \xrightarrow[\lambda\to 0]{} \int \Phi(\Theta f)\Phi(f) d\mu(\Phi).$$

□

# 8 Reflection positivity for the Laplacian covariance

Theorem 7.1 reduces the reflection positivity of $L^2(\mathscr{H}', d\mu)$. to the reflection positivity of a bilinear form $C$ on $\mathscr{H}$. In this section, we will construct a reflection positive $C$ of great physical interest:

**Definition 8.1** (Resolvent of the Laplace-Beltrami operator)**.** *Let $\Delta = \sum_{i=1}^{d} \partial_{x_i}^2$ be the **Laplace operator**, which is an essentially self-adjoint operator on $C_0^\infty(\mathbb{R}^d)$. Then the resolvent $C = (-\Delta + m^2)^{-1}$ is a bounded operator on $L^2(\mathbb{R}^d)$ that gives the following bilinear form on $L^2$:*

$$\langle f, Cg\rangle_{L^2} \equiv C(f,g) \equiv \int d\vec{x}\, d\vec{y} \bar{f}(\vec{x}) C(\vec{x},\vec{y}) g(\vec{y}), \qquad C = (-\Delta + m^2)^{-1}.$$

*$C$ is the Green's function of the equation $\Delta\phi = m^2\phi$. This equation is of physical significance because it is the analytic continuation to imaginary time of the equation of motion for a free particle (a particle that does not interact).*

*More generally, given a $d$-dimensional Riemannian manifold $M$ with Levi-Civita connection $\nabla$, then let $\Delta_M \equiv \nabla^*\nabla$ denote the (negative-definite) covariant* ***Laplace-Beltrami operator****. If $M$ is complete then $\Delta_M$ is essentially self-adjoint as an operator on functions, forms, and tensors [39, Theorem 2.4]. Then, by the spectral theorem, the resolvent $C_M = (-\Delta_M + m^2)^{-1}$ is a well-defined bounded operator that gives the inner product*

$$\langle f, C_M g\rangle_{L^2(M)} \tag{11}$$

In Theorem 8.3, we prove that the inner product of Equation 11 is reflection positive when $\Delta_M$ is the Laplace-Beltrami operator on any complete Riemannian manifold that admits a reflection. Then, in Theorem 8.4, we prove that on $\mathbb{R}^d$ reflection positivity still holds when Dirichlet and Neumann boundary conditions are applied to the Laplacian.

Let $M$ be a $d$-dimensional, complete, connected Riemannian manifold, let $G \equiv \mathrm{Isom}(M)$ be the isometry group of the manifold, and let $\Theta \in G$ be a reflection in the sense of Definition 4.7. Suppose that $\Theta$ is dissecting, meaning that it partitions the manifold as $\Omega_- \sqcup \Sigma \sqcup \Omega_+$ where the reflection hyperplane $\Sigma$ disconnects its complement. Note that in this section we do not assume that $M$ is static, although the reflections that we defined for a static manifold are dissecting. Let $U$ denote the unitary representation of $G$ on $L^2(M)$, defined by $U_\psi f \equiv f^\psi \equiv f \circ \psi^{-1}$ for $\psi \in G$. Let $\Delta_M$ denote the Laplace-Beltrami operator. This operator is an isometry invariant, which can be checked by direct calculation on the coordinate expression:

$$\Delta_M u = -\frac{1}{\sqrt{g}} \sum_{j,k=1}^{d} \partial_j (g^{jk}\sqrt{g}\partial_k u).$$

Then the following lemma gives us that $[U_\psi, C] = 0$, i.e. $Cf^\psi = (Cf)^\psi$:

**Lemma 8.2.** *Let $T$ be an operator on a Banach space $X$ such that the resolvent set of $T$ is non-empty. In order that $T$ commute with a bounded operator $A$ on $X$, it is necessary that $R(z,T) = (T-z)^{-1}$ commute with $A$.*

*Proof.* See [29, Theorem 3.6.5]. □

We then have the following theorem due to [25]:

**Theorem 8.3** (Reflection Positivity for Laplace-Beltrami)**.** *Let $M$ be a complete, connected Riemannian manifold with a dissecting reflection $\Theta$ in the sense discussed above. Let $\Omega_+ \sqcup \Sigma \sqcup \Omega_-$ denote the partition of the manifold by the reflection hyperplane. For all $f \in C_0^2(\Omega_+)$,*

$$0 \le \langle f^\Theta, Cf\rangle_{L^2}. \tag{12}$$

*Proof.* For convenience, let $u = Cf$. Then

$$\langle u, f^{\Theta}\rangle_{L^2} = \langle u, C^{-1}u^{\Theta}\rangle_{L^2} = \int_{\Omega_-} \overline{u} C^{-1} u^{\Theta} dV$$
$$= \int_{\Omega_-} \overline{u} C^{-1} u^{\Theta} dV - \int_{\Omega_-} \overline{C^{-1}u} u^{\Theta} dV.$$

The first line uses the fact that $[U_{\Theta}, C] = 0$ and that $f^{\Theta}$ has support only on $\Omega_-$. The second line uses the fact that $C^{-1}u = f$ is zero on $\Omega_-$. Replacing $C^{-1}$ with $(-\Delta_M + m^2)$ and integrating by parts, we find

$$\langle f^{\Theta}, u\rangle_{L^2} = \int_{\Sigma} [u^{\Theta} \nabla_n \bar{u} - \bar{u} \nabla_n u^{\Theta}] dS,$$

where $n$ is the normal vector to $\partial\Omega$. By construction (see Proposition 4.8), we have that for $p \in \partial\Omega$, $d\Theta_p = d\Theta_p^{-1} = \mathrm{diag}(-1, 1, \dots, 1)$ in a coordinate basis where the first coordinate is in the direction of $n_p$. Using $(\nabla_n u^{\Theta})_p = -(\nabla_n u)_p$ and $u^{\Theta} = u$ on $\Sigma$, the above equation simplifies to

$$\langle f^{\Theta}, Cf\rangle_{L^2} = 2\,\mathrm{Re}\left[\int_{\Sigma} u \nabla_n \bar{u} dS\right].$$

Now we show by manipulation that the quantity in brackets is real and positive:

$$\int_{\partial\Omega} u n_a \nabla^a \bar{u} dS = \int_{\Omega_-} \nabla_a (u \nabla^a \bar{u}) dV$$
$$= \int_{\Omega_-} (\nabla_a u \nabla^a \bar{u} + u \Delta \bar{u}) dV$$
$$= \int_{\Omega_-} (|\nabla u|^2 + m^2 |u|^2) dV \geq 0.$$

The last equality comes from the fact that $\Delta u = m^2$ in $\Omega_-$, which holds because $f$ is supported in $\Omega_+$. □

Now we restrict our attention to the setting $M = \mathbb{R}^d$. We let $\Theta$ denote a reflection on $\mathbb{R}^d$, as well as the action of that reflection on $L^2(\mathbb{R}^d)$. We prove the result, due to Glimm and Jaffe [15], that reflection positivity still holds in the presence of boundary conditions on the Laplacian. In this discussion, we use the following notion of inequality for bilinear forms: $A \leq B$ means that $\mathrm{Dom}(B) \subset \mathrm{Dom}(A)$ (form domains) and $\langle x, Ax\rangle \leq \langle x, Bx\rangle$ for all $x \in \mathrm{Dom}(B)$.

**Theorem 8.4** (Reflection Positivity for Laplacian with boundary conditions)**.** *Suppose that $\Delta_B$ is the Laplacian on $\mathbb{R}^d$ with boundary data $B$ on a finite union $\Gamma$ of piecewise smooth hypersurfaces. Suppose that the boundary data consists of a mixture of Dirichlet and/or Neumann conditions, and suppose that the boundary conditions are symmetric under the reflection $\Theta$. Let $C = (\Delta_B + m^2)^{-1}$. Then $C$ is reflection positive with respect to $\Theta$.*

*Note that this implies theorem 8.3 in the case that $M = \mathbb{R}^d$.*

*Proof.* By the fact that the commutator $[\Delta, \Theta] = 0$ and the fact that $\Theta$ preserves the boundary conditions, we have that $[\Delta_B, \Theta] = 0$. Then Lemma 8.2 implies that $[C, \Theta] = 0$.

Let $\Sigma \subset \mathbb{R}^d$ denote the plane of reflection that decomposes $\mathbb{R}^d$ into $\mathbb{R}^d_- \sqcup \Sigma \sqcup \mathbb{R}^d_+$. Let $\Pi_\pm$ be operators on $L^2(\mathbb{R}^d)$ that give orthogonal projection onto $\Omega_\pm \equiv L^2(\mathbb{R}^d_\pm)$. Then we wish to prove that

$$\Pi_+ \Theta C \Pi_+ \geq 0, \tag{13}$$

as a bilinear form on $L^2(\mathbb{R}^d)$.

We can write this positivity condition as

$$\Pi_+[C - (I - \Theta)C]\Pi_+ \geq 0. \tag{14}$$

First, we will prove that $(I - \Theta)C$ restricted to $\Omega_\pm$ is equal to $C_D = (\Delta'_B + I)^{-1}$ where $\Delta'_B$ is the Laplacian operator with Dirichlet data on $\Sigma$ and $B$ data on $\Gamma \setminus \Sigma$. Second, we will prove that $C_D \leq C$.

The first claim comes from the fact that $(I - \Theta)C(x, y)$ vanishes on $\Sigma$, and $(-\Delta_B + I)(I - \Theta)C(x, y) = \delta(x - y)$ on $\mathbb{R}^d_\pm$, i.e. $(I - \Theta)C$ satisfies the characteristic differential equation of $C_D$.

The second claim follows from the fact that $-\Delta_B \leq -\Delta'_B$ as bilinear forms because $\Delta'_B$ is defined as the restriction of $\Delta_B$ to functions vanishing on $\Sigma$. Taking inverses gives the desired result. □

**Remark 8.5.** *The positivity condition in equation 13 could alternatively be written as*

$$\Pi_+[(I + \Theta)C - C]\Pi_+ \geq 0, \tag{15}$$

*and the proof could be based on this fact by proving:*

1. *$(I + \Theta)C$ restricted to $\Omega_\pm$ is equal to $C_N = (\Delta'_B + I)^{-1}$ where $\Delta'_B$ is the Laplacian operator with Neumann data on $\Sigma$ and $B$ data on $\Gamma \setminus \Sigma$.*

2. *$C \leq C_N$ as bilinear forms.*

Let $\Delta$ be the free Laplacian with covariance $C$. In the proof of the previous theorem, we discussed the monotonicity of bilinear forms $C_D \leq C \leq C_N$ where $C_D$ (resp. $C_N$) corresponds to the free Laplacian with Dirichlet (resp. Neumann) conditions on $\Sigma$. This monotonicity can be proven from the coordinate representations of the covariance forms. An analogous result for general Riemannian manifolds is proven in [25] using Theorem 8.3

# 9 Reflection positivity for the Dirac covariance

The entire paper up until now presents a unified discussion: we construct a Euclidean space $\mathcal{E}$ over $M$ equipped with a reflection positive bilinear form, and we use this reflection positivity to analytically continue the theory to a Lorentzian setting. The final step of our argument, carried out in the previous section, was to prove the reflection positivity for the resolvent $C$ of the Laplace-Beltrami operator. We found earlier that the reflection positivity of $C$ implies the reflection positivity of $\mathcal{E} \equiv L^2(\mathscr{H}', d\mu_C)$.

In this final section of the paper, we branch out from the preceding argument and prove the reflection positivity of a related bilinear form over $M$ that does not directly give a measure. This form can instead be used to define a Berezin integral [2] and to develop the the physical theory of fermionic particles. This is more than we will undertake in this paper. Our purpose in this section is to illustrate the rich array of reflection positive forms over the Riemannian manifold $M$.

In this section, we prove the reflection positivity of $C = (D - m)^{-1}$ where $D$ is a **Dirac operator**. Dirac operators on Riemannian manifolds were introduced by Atiyah and Singer for their 1963 proof of the Atiyah-Singer Index Theorem. This was a rediscovery of a physically significant idea of Dirac. Working in 4 dimensional Minkowski spacetime, Dirac sought an operator $D$ such that $D^2$ equals the Laplacian. This operator is given by $\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$, where $\gamma^\mu$ denotes the Gamma matrices. The physics of electron motion in Minkowski space is famously described by the Dirac equation:

$$(-i\partial\!\!\!/ + m)\psi = 0.$$

In the following section, we generalize the definition of the Dirac operator to the setting of Riemannian manifolds. In section 9.2, we prove the desired reflection positivity result, which is due to [25].

## 9.1 Introduction to the Dirac operator

To generalize the theory of this operator $\partial\!\!\!/$, we ask the following: Let $M$ be a Riemannian manifold and $E$ a vector bundle over $M$. Let $\Gamma(M, E)$ denote the space of smooth sections. What are the first-order differential operators $D$ that square to the generalized Laplacian on $\Gamma(M, E)$? Write

$$D = \sum_k a^k(x)\partial_k + b(x),$$

where $a^k(x)$ and $b(x)$ are sections of $\mathrm{End}(E)$. Squaring this and comparing it to the generalized Laplacian, we find that $D^2$ is the generalized Laplacian if and only if for $u, v \in T_x^*M$, we have

$$\langle a(x), u\rangle\langle a(x), v\rangle + \langle a(x), v\rangle\langle a(x), u\rangle = -2\langle u, v\rangle_x$$

where $\langle\cdot,\cdot\rangle_x$ is the inner product on $T_x^*M$ that arises from the metric on $M$. The above equation is the defining relation for the Clifford algebra of $T_x^*M$. This leads to the following definition for a Dirac operator:

Let $M$ be a Riemannian manifold. Let $\mathrm{C}l(M)$ denote the **Clifford bundle** of $M$. This is the bundle over $M$ whose fiber at $x \in M$ is the Clifford algebra of $T_x^*M$ with metric as inner product. These fibers are denoted $\mathrm{C}l_x(M)$. Recall, by generalities of Clifford algebras, that there is a natural inclusion $T_x^*(M) \hookrightarrow \mathrm{C}l_x(M)$.

Let $E$ denote a Hermitian vector bundle over $M$ such that each fiber $E_x$ is a self-adjoint $\mathrm{C}l_x(M)$ module in a smooth fashion. The inclusion $T_x^*(M) \hookrightarrow \mathrm{C}l_x(M)$ gives rises to a bundle map $\mathbf{m} : T^*(M) \otimes E \to E$ called **Clifford multiplication**. For convenience, we denote Clifford multiplication as $\xi \cdot v$.

If $\nabla$ is a connection for $E \to M$, then we have a sequence

$$\Gamma(M, E) \xrightarrow{\nabla} \Gamma(M, T^*(M) \otimes E) \xrightarrow{\mathbf{m}} \Gamma(M, E).$$

**Definition 9.1** (Dirac operator)**.** *Given a Riemannian manifold $M$ and a Hermitian vector bundle $E \to M$ satisfying the above conditions, the Dirac operator $\partial\!\!\!/$ is the composition of the above maps, i.e. $\partial\!\!\!/ = \mathbf{m}\nabla : \Gamma(M, E) \to \Gamma(M, E)$.*

A Dirac operator can be described in local coordinates. Let $O \subset M$ be an open subset with an orthonormal frame $\{e_j\}$ of tangent vector fields, and let $\{v_j\}$ denote a dual frame of 1-forms. For $\phi \in \Gamma(M, E; O)$:

$$\partial\!\!\!/\phi = \sum_j v_j \cdot \nabla_{e_j}\phi. \tag{16}$$

A **Clifford connection** on $E$ is a metric connection $\nabla$ that is compatible with Clifford multiplication in the sense that

$$\nabla_X(\xi \cdot v) = (\nabla_X \xi) \cdot v + \xi \cdot \nabla_X v$$

for a vector-field $X$, a 1-form $\xi$, and a section $v$ of $E$. Here $\nabla_X \xi$ arises from the Levi-Civita connection on $M$. A Clifford connection has the powerful property that the associated Dirac operator is skew-symmetric with respect to the inner product of $L^2(E)$ given by

$$\langle u, v\rangle = \int_M \langle u(x), v(x)\rangle_x \, dV.$$

**Proposition 9.2.** *If $\nabla$ is a Clifford connection on $E$ then $i\partial\!\!\!/$ is symmetric.*

*Proof.* Let $\alpha, \beta$ be smooth sections of compact support in $\Gamma(M, E)$. We wish to prove that

$$\int_M \left[\langle i\partial\!\!\!/\alpha, \beta\rangle - \langle \alpha, i\partial\!\!\!/\beta\rangle\right] dV = 0.$$

It is sufficient to handle the case that $\alpha, \beta$ have support in a compact set $U$ with a local orthonormal frame $e_j$ for smooth vector fields, and a dual orthonormal frame $v_j$ for 1-forms. On this set, we have

$$\partial\!\!\!/\alpha = \sum v_j \cdot \nabla_{e_j}\alpha.$$

Now define a vector field $X$ on $U$ by $\langle X, v\rangle = \langle \alpha, v\cdot\beta\rangle$ for $v \in \wedge^1 U$. The desired result will follow from the divergence theorem if we can prove

$$\operatorname{div} X = \left[\langle \partial\!\!\!/ \alpha, \beta\rangle + \langle \alpha, \partial\!\!\!/ \beta\rangle\right].$$

By definition

$$\operatorname{div} X = \sum_j \langle \nabla_{e_j} X, v_j\rangle.$$

Using the fact that $\nabla$ is a metric connection and the definition of $X$, we have

$$\operatorname{div} X = \sum_j \left[e_j \cdot \langle X, v_j\rangle - \langle X, \nabla_{e_j} v_j\rangle\right] = \sum_j \left[e_j \cdot \langle \phi, v_j\cdot\psi\rangle - \langle \phi, (\nabla_{e_j} v_j)\cdot\psi\rangle\right].$$

Expanding and using the fact that $\nabla$ is a Clifford connection, this becomes

$$\operatorname{div} X = \sum_j \left[\langle \nabla_{e_j}\phi, v_j\cdot\psi\rangle + \langle \phi, v_j\cdot\nabla_{e_j}\psi\rangle\right],$$

as desired. □

To give a reflection positivity result, we need to introduce a condition on $M$ that enhances the symmetry of $i\partial\!\!\!/$ to essential self-adjointness on smooth sections of compact support. Such a condition is proven in [6], and we sketch it here. For any first-order differential operator $L$, let $\sigma(\nu, x)$ denote its symbol. For each $x \in M$ and $\nu \in T_x^*M$, the symbol is a linear map $\sigma(\nu, x) : E_x \to E_x$ given by

$$\sigma(\nu, x)e = L(gf)(x) - g(x)(Lf)(x),$$

where $f \in \Gamma(M, E)$ is a section such that $f(x) = e$, and $g \in C^\infty(M)$ is a function with $dg_x = \nu$. We wish to globalize this solution. When $L$ is symmetric, then standard results give existence, uniqueness, and smoothness for the solutions of the hyperbolic system $\partial u/\partial t = Lu$. Define the local propagation velocity of the system in the following way:

$$c(x) = \sup\{\|\sigma(\nu, x)\| : \nu \in T_x^*M, |\nu| = 1\},$$

where $\|\cdot\|$ is the operator norm on $E_x$. Then for $\Omega \subset M$, we define $c(\Omega) = \sup\{c(x)\colon x \in \Omega\}$. Then, define

$$c(r) = c(S_r),$$

where $S_r$ is the ball of radius $r$ about an arbitrary reference point $x_0 \in M$. Then [6, Theorem 2.2] gives that $L$ is essentially self-adjoint if the following two conditions hold:

1. $M$ is complete

2. $\int_0^\infty dr/c(r) = \infty$.

Thus, if the manifold $M$ satisfies these two conditions, the operator $i\partial\!\!\!/$ is essentially self-adjoint. This will be an assumption in the reflection-positivity theorem of the next section.

Note that our definition of the Dirac operator and our discussion of its properties includes the most prominent example as a special case:

**Remark 9.3** (Spinor bundle)**.** *Let $M$ be a complete $n$-dimensional oriented Riemannian manifold. Associated with $M$ is the bundle $P \to M$ of orthonormal frames. It is a principal* $\mathrm{SO}(n)$ *bundle. A* ***spin structure*** *on $M$ is a lift $\tilde{P} \to M$ to a principal* $\mathrm{Spin}(n)$*-bundle such that $\tilde{P}$ is a double-covering characterized in the following way: the action of* $\mathrm{Spin}(n)$ *on the fibers of $\tilde{P}$ is compatible with the action of* $\mathrm{SO}(n)$ *on the fibers of $P$ via the covering homomorphism* $\mathrm{Spin}(n) \to \mathrm{SO}(n)$*. When such a structure exists, there is a natural associated Clifford bundle called the* ***spinor bundle****, which has a Clifford connection. Details of this construction can be found in [3, 40]. The Dirac operator $i\partial\!\!\!/$ is essentially self-adjoint for this connection, and the theory that we are developing applies [6, Section 3]*

## 9.2 Proof of reflection positivity

Let $M$ be a complete static Riemannian manifold. Let $x_i$ give local coordinates such that $\frac{\partial}{\partial x_0}$ is the hypersurface-orthogonal Killing field that gives time-translation. We let $t = x_0$. Locally the metric takes the form

$$ds^2 = F(x)dt^2 + G_{jk}(x)dx^j dx^k.$$

Let $\Theta$ be the reflection map around the time-zero surface $\Sigma$. Decompose $M = \Omega_- \sqcup \Sigma \sqcup \Omega_+$. Let $E \to M$ be a holomorphic Clifford bundle with Clifford connection $\nabla$. Let $\Theta^*$ denote the pullback of $\Theta$ that acts on sections of $E$. Let $dx^i$ denote the local frame of one-forms, and let $\gamma^i$ denote the Clifford multiplication by $dx^i$, i.e. $\gamma^i(v) = dx_i \cdot v$. Then the anti-commutator obeys the relation:

$$\{\gamma^i, \gamma^j\} = 2g^{ij}\,\mathrm{id},$$

where $g^{ij}$ is the inverse metric. On a static manifold, the operator $\gamma^0$ has a coordinate free meaning. Then, a simple calculation gives:

$$\{\gamma^0\Theta^*, \partial\!\!\!/\} = \left\{\gamma^0\Theta^*, \sum_j \gamma^j \nabla_{e_j}\right\} = 0.$$

We have the following theorem due to [25]:[3]

[3]The current arXiv version of this paper (version 2) incorrectly assumes that $i\partial\!\!\!/$ is self-adjoint for all complete $M$. The published paper includes the condition on $c(r)$.

**Theorem 9.4.** *Let $M$ be a complete, static Riemannian manifold with the structure described above. Let $E \to M$ be a holomorphic Clifford bundle with Clifford connection $\nabla$, and let $\gamma^i$ be as defined above. Suppose that $\int_0^\infty dr/c(r) = \infty$ in the sense of the previous section, so that $i\partial\!\!\!/$ is essentially self-adjoint. Then for all smooth sections $\phi$ of compact support in $\Omega_+$, we have*

$$\langle \gamma^0 \Theta^* \phi, (\partial\!\!\!/ - m)^{-1}\phi \rangle,$$

*where this is the inner product in $L^2(E)$.*

*Proof.* For convenience, let $\theta \equiv \gamma^0\Theta^*$ and $v = (\partial\!\!\!/ - m)^{-1}\phi$. Let $A(\phi) = \langle \gamma^0\Theta^*\phi, (\partial\!\!\!/ - m)^{-1}\phi\rangle$, which we wish to prove is positive. Then

$$A(\phi) = \langle \theta\partial\!\!\!/ v, v\rangle - m\langle \theta v, v\rangle = -\int_{\Omega_-} \left[\langle \partial\!\!\!/ \theta v, v\rangle + m\langle \theta v, v\rangle\right].$$

Where the second equality comes from the fact that $\{\theta, \partial\!\!\!/\} = 0$. Then, using the fact that $(\partial\!\!\!/ - m)u = \phi = 0$ on $\Omega_-$, we have

$$A(\phi) = -\int_{\Omega_-} \left[\langle \partial\!\!\!/ \theta v, v\rangle + m\langle \theta v, v\rangle + \langle \theta u, (\partial\!\!\!/ - m)u\rangle\right] = \int_{\Omega_-} \left[\langle \partial\!\!\!/ \theta u, u\rangle - \langle \theta u, \partial\!\!\!/ u\rangle\right].$$

We saw in the course of proving Proposition 9.2 that for smooth sections $\alpha, \beta$, we have

$$\operatorname{div} X = \left[\langle \partial\!\!\!/ \alpha, \beta\rangle + \langle \alpha, \partial\!\!\!/ \beta\rangle\right],$$

where $X$ is the vector field defined by $\langle X, v\rangle = \langle \alpha, v \cdot \beta\rangle$ for $v \in \wedge^1 M$. Taking $\alpha = \theta v$ and $\beta = v$, we have

$$A(\phi) = \int_{\Omega_-} \operatorname{div} X \, dV,$$

where $\langle X, v\rangle = \langle \theta v, v \cdot u\rangle_E$.

Let $\hat{n} \equiv F^{-1/2}\frac{\partial}{\partial x_0}$ denote the unit normal vector of $\Sigma$ that points into $\Omega_-$. Let $\nu = F^{1/2}dx_0$ denote the dual one form. The divergence theorem gives

$$A(\phi) = \int_{\Omega_-} \operatorname{div} X dV = \int_\Sigma \langle X, \nu\rangle dS.$$

On $\Sigma$, we have

$$\langle X, \nu\rangle = \langle \theta u, \nu \cdot u\rangle_E = \langle \gamma^0(u), \sqrt{F}\gamma^0(u)\rangle.$$

Thus

$$A(\phi) = \int_\Sigma \langle \gamma^0(u), \sqrt{F}\gamma^0(u)\rangle dS \geq 0.$$

□

# 10 Conclusion

In this paper, we have constructed a theory of reflection positive Euclidean fields for all complete, static Riemannian manifolds $M$ with compact spatial hypersurfaces. This reflection positivity allows us to analytically continue the Euclidean theory to a Lorentzian theory. The representations of $G^0_{\mathrm{lor}}$ that are constructed in this way are at the heart of mathematical quantum field theory. This exposition provides the rigorous foundation for studying physics at imaginary time. The techniques described here underlie the foremost efforts to construct a mathematical theory of particles.

Reflection positivity, as defined in the present paper, has applications throughout physics and mathematics. In the final section, we used the example of the Dirac operator to illustrate the richness of the reflection positive forms available over a Riemannian manifold. Our treatment of reflection positivity is sufficient to understand its use by A. Jaffe et al. to construct representations of the Heisenberg algebra on a Riemann surface [22] and its use by V. Pestun to compute the partition function for supersymmetric Yang-Mills on the four-sphere [36]. Surveys of material outside the scope of this exposition include M. Biskup's paper on reflection positivity in statistical mechanics [4] and P. Jorgensen and G. Ólafsson's papers on reflection positivity in representation theory [27, 28]. The latter, in particular, are excellent further reading for mathematicians.

# 11 Appendix A: Cited theorems

## Bochner-Minlos theorem

This theorem characterizes a measure $d\mu$ on the continuous dual of a nuclear space. It is due to Salomon Bochner and Robert Minlos. See, for instance, [13, IV.4]:

**Theorem 11.1** (Bochner-Minlos)**.** *Suppose that $V$ is a nuclear space and $V'$ is its continuous dual. If $d\mu$ is a regular Borel measure on $V'$ with total weight $1$, then its generating functional $S = \int e^{i\Phi(f)} d\mu$ satisfies the following properties:*

1. *Continuity (in the Fréchet topology of $V$)*
2. *Positive definiteness: for all $c_i \in \mathbb{C}$, $f_i \in V$,*

$$0 \leq \sum_{i,j=1}^{N} \bar{c}_i c_j S(f_i - f_j).$$

3. *Normalization: $S(0) = 1$.*

*Conversely, given a functional $S$ defined on $V$ and satisfying the previous three properties, then $S$ is the inverse Fourier transform of a unique regular Borel measure $d\mu$ with normalization $\int d\mu = 1$.*

### Stone's theorem

In the mathematical physics literature, the name "Stone's theorem" is abused to refer to several theorems that imply the existence of self-adjoint generators for one-parameter semigroups. The eponymous theorem is:

**Theorem 11.2** (Stone's theorem)**.** *Let $U(t)$ be a strongly continuous one-parameter unitary group on the Hilbert space $\mathcal{H}$. Then there is a unique self-adjoint operator $A$ on $\mathcal{H}$ such that $U(t) = e^{itA}$ for $t \in \mathbb{R}$.*

A reference is [37, Theorem 6.2]. Stone's theorem is used in the current exposition to prove Theorem 6.7.

Another theorem that sometimes goes under the same name is:

**Theorem 11.3.** *Let $T(t)$ be a contraction semigroup of self-adjoint operators on the Hilbert space $\mathcal{H}$. Then there exists a unique positive self-adjoint operator $A$ on $\mathcal{H}$ such that $T(t) = e^{-tA}$ for $t \geq 0$.*

A reference is [37, Proposition 6.14]. This result is used in the current exposition to prove Theorems 5.10 and 5.12.

The theory of symmetric local semigroups described in Section 6.2 gives another set of conditions under which a semigroup has a self-adjoint generator.

## 12 Acknowledgments

Thank you to Professor Arthur Jaffe for the guidance in the three years that I have known him at Harvard, for his constant assistance in exploring mathematical physics, and for his support during the process of writing this thesis. Thank you to the other members of Professor Jaffe's Fall 2012 reading course in mathematical field theory: Dmitri Gekhtman, Nikko Pomata, Clay Cordova, Alex Lupasca, and Roberto Martinez. Thank you to Vasily Pestun for his discussion regarding [36]. Thank you to my family: my mom, my dad, Catherine, and Thayer.

Thank you to my mathematics and physics professors at Harvard. This thesis has benefited from every course that I took with them. In the mathematics department they are Noam Elkies, Joe Harris, Michael Hopkins, Peter Kronheimer, Curtis McMullen, Wilfried Schmid, and Shlomo Sternberg. In the physics department they are Howard Georgi, Arthur Jaffe, Erel Levine, Misha Lukin, Matthew Schwartz, Andrew Strominger, and Cumrun Vafa. Thank you to Steve Carlip of UC Davis, Erel Levine of Harvard, and Sergei Tabachnikov of Penn State, who have advised me in research projects. Thank you to my academic advisers Peter Kronheimer, David Morin, and Clifford Taubes for their support.

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