QUALS Collection

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Harvard University Mathematics Department

The Qualifying Exam in Mathematics

The purpose of the Qualifying Exam

One of the requirements for a doctorate in Mathematics at Harvard is a familiarity with a reasonably broad sweep of the science of the mathematics. The sole use of the Qualifying Examination is to measure the breadth of a student's mathematical knowledge.

The mechanics of the Qualifying Exam

The exam is given twice a year, usually in late September and in early February. It consists of three 3-hour exams held on consecutive afternoons, containing a total of 18 problems. A score of approximately 100 out of 180 is usually a passing grade, but "conditional passes" are given when the break-down of the score indicates that the student is primarily weak in one area. In that case, he or she must pass an oral exam in their weak subject within the next 6 months with a specified professor.

A student may take the exam any number of times, starting from the September exam of the student's first semester. A student is not penalized in any way for failing the exam once or several times, but, students are encouraged to pass the exam by the end of the second year in residence, in order to devote time to honest mathematical research.

Preparing for the Qualifying Exam

The Department offers a basic sequence of mathematics courses for the first four semesters in residence; and the successful completion of this sequence plus minimal memory skills should amply prepare the student for the Qualifying Exam. The basic courses are

Math 212a,b	(real analysis)
Math 213a,b	(complex analysis)
Math 230a,b	(differential geometry)
Math 250a,b	(algebra)
Math 260a,b	(algebraic geometry)
Math 272a,b	(topology)

These courses cover substantially more mathematics than the Qualifying Exam requires; a student who passes the exam upon entrance will also find these courses interesting.

Many students in the past have studied individually, or in small groups for the Qualifying Exam. This is encouraged and, as a guide to the exam, copies of old exams are made available in the Department office. It is recommended that a student work through several of these carefully to prepare for the exam.

Now a word of warning: Do not let studying for the Qualifying Exam interfere with the time spent in the basic courses. In the long run, you will learn far more mathematics from the basic courses than from studying for the Qualifying Exam. It is for this reason that the Qualifying Exams are held at the start of the semester. If cramming is your compulsion, you can satisfy it in the summer, or in January's reading period.

Perspective

For most people, graduate school is not easy – it is a time of trauma and self-doubt. So, it is important to maintain the proper perspective. By the end of your incarnation as a graduate student, you will have learned a tremendous amount of mathematics. This knowledge will serve you throughout your life. By comparison, the Qualifying Exam is a check mark in your file; as such, it is meaningless to you as soon as you pass it.

Syllabus

What follows is a rough outline of the topics on which questions might be asked, including recommended books and courses:

A. ALGEBRA.

Thorough understanding of linear algebra, elementary theory of finite groups and their representations, rings and fields, and Galois theory. Reasonable familiarity with commutative algebra (ideal theory in Noetherian and Dedekind rings, notion of integral dependence and integral closure, modules) and its applications to polynomial rings and rings of algebraic integers.

Math 122, 123, 250a,b cover all this.

References:
Jacobson, Basic Algebra I, II
Herstein, Topics in Algebra
Van der Waerden, Algebra, I, II
Zariski-Samuel, Commutative Algebra, I
Atiyah-MacDonald, Commutative Algebra
Lang, Algebra

B. ALGEBRAIC TOPOLOGY.

Knowledge of the classification of surfaces, covering spaces and fundamental group, of homology and cohomology groups and elementary aspects of homotopy theory. This includes ability to calculate homology and homotopy groups, degree of a map, etc. in simple situations and familiarity with basic topological spaces, e.g. real and complex projective space, Lie groups.

Math 131, 272a cover this material.

References:
Greenberg, Lectures in Algebraic Topology
Spanier, Algebraic Topology, Ch. I-V
Massey, Algebraic Topology, An Introduction

C. DIFFERENTIAL GEOMETRY.

Manifolds and the calculus of forms and vector fields on them. The classical theory of curves and surfaces in R^3 (Frenet theory, Gaussian curvature, geodesics, the Gauss-Bonnet theorem). Introductory theory of Riemannian manifolds. Computational techniques are essential.

Math 25, 136, 230a,b cover this material.

References:
O'Neil, Elementary Differential Geometry, last chapter
DoCarmo, Differential Geometry of Curves and Surfaces
Guilleman & Pollock, Differential Topology

D. ALGEBRAIC GEOMETRY.

Projective space of a field; affine and projective varieties and their ideals, coordinate rings and function fields; regular and rational maps. Concepts of irreducibility, dimension, smoothness and singularity, Zariski tangent space and degree of projective variety; Bezout's theorem.

Math 260a,b cover this material.

References:
Hartshorne, Algebraic Geometry, Chapter 1 only.
Shafarevich, Basic Algebraic Geometry, Chapters 1 and 2 only.
Fulton, Algebraic Curves
Kendig, Elementary Algebraic Geometry

E. ANALYSIS

1. Point set topology, including metric spaces.

2. Measure and integration on general measure spaces, including Fubini's theorem and the Radon-Nikodym theorem.

3. Basic facts about Banach and Hilbert spaces, including the spectral theorem for bounded self-adjoint operators, and applications to Fourier series, Fourier integrals and differential equations. Also, contraction mapping theorem for Banach spaces.

Math 212a,b , in general, covers this material.

References: Royden, Real Analysis *good for A & B.*
Rudin, Functional Analysis *good for C.*

4. Ordinary Differential Equations: existence and uniqueness of solutions, linear equations, behavior of solutions near equilibrium points, regular singular points.

References: Birkhoff & Rota, Ordinary Differential Equations, Chapters 1-6; 9

5. Partial Differential Equations with constant coefficients, separation of variables, eigenvalue equations, Fourier series.

Reference: Churchill & Brown, Fourier Series and Boundary Value Problems

F. COMPLEX ANALYSIS.

Math 213 will cover this material.

References: Ahlfors, Complex Analysis
Cartan, Complex Analysis

G. CALCULUS!

Every qualifying exam in recent years has had at least one calculus problem and many students have not done well.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday, March 5, 2002 (Day 1)

Each question is worth 10 points, and parts of questions are of equal weight.

1a. Describe, as a direct sum of cyclic groups, the cokernel of the map $\varphi : \mathbf{Z}^3 \to \mathbf{Z}^3$ given by left multiplication by the matrix

$$\begin{bmatrix} 15 & 6 & 9 \\ 6 & 6 & 6 \\ -3 & -12 & -12 \end{bmatrix}.$$

2a. Show that any smooth projective curve of genus zero over a field k is isomorphic to a plane conic over k.

3a. Let f be a function that is analytic on the annulus $1 \leq |z| \leq 2$ and assume that $|f(z)|$ is constant on each circle of the boundary of the annulus. Show that f can be meromorphically continued to $\mathbf{C} - \{0\}$.

4a. Give an example of a complete smooth algebraic curve C of degree four in $\mathbf{P}^2_{\mathbf{C}}$, then find the dimension of the space of holomorphic forms on C.

5a. Let Σ be a closed 2–dimensional Riemannian manifold whose sectional curvature is everywhere negative. Show that there is no isometric immersion of Σ into Euclidean space $\mathbf{R}^3$.

6a. A pair of (nonzero) points P and Q in $\mathbf{R}^3$ are called **antipodal** if $Q = -P$. By the two sphere $S^2 \subset \mathbf{R}^3$ we mean, as usual, the topological subspace $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbf{R}^3$. Let $f : S^2 \to \mathbf{R}^2$ be a continuous map. Show that there is at least one pair of antipodal points $\{P, Q\}$ on the two sphere such that

$$f(P) = f(Q).$$

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday, March 6, 2002 (Day 2)

Each question is worth 10 points, and parts of questions are of equal weight.

1b. Let H be a Banach space. Show that there are no bounded operators A, B on H for which

$$AB - BA = I$$

Hint: show for any positive integer n, that

$$AB^n - B^n A = nB^{n-1}$$

On the other hand, for H some space of functions on $\mathbf{R}$, let A be the operator of differentiation $\frac{d}{dx}$ and let B be the operator of multiplication by x. Then

$$AB - BA = I.$$

What is going on here?

2b. Show that the sum $\sum 1/p$, where p ranges over all prime numbers, does not converge.

3b. For which positive integers n is $\mathbf{Z}^n$ a set-theoretic union of finitely many proper subgroups? (Here $\mathbf{Z}^n$ is the direct sum of n copies of $\mathbf{Z}$.)

4b. Suppose we have a function regular on a closed disk. Show that its absolute value at the center of the disk does not exceed the arithmetic mean of its absolute value on the boundary of the disk.

5b. Let $R = \mathbf{Z}[x]/(f)$, where $f = x^4+42x^3-11x^2+22x-2002002002002002$, and let $I = 3R$ be the principal ideal of R generated by 3. Find all prime ideals of R that contain I. (Give generators for each.)

6b. Let G be a nonabelian group of order 16 containing an element of order 8. Give the character table of G.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday March 7, 2002 (Day 3)

Each question is worth 10 points, and parts of questions are of equal weight.

1c. Let Q be a nonsingular quadratic surface in $\mathbf{P}^3_{\mathbf{C}}$. Prove that any point $p \in Q$ is the intersection of two lines on Q.

2c. Let $X \subset \mathbf{P}^n_{\mathbf{C}}$ be an l-dimensional projective variety. Denote by $\mathbf{G}$ the Grassmannian of k-planes in $\mathbf{P}^n_{\mathbf{C}}$, and let $Z \subset \mathbf{G}$ be the subset of k-planes meeting X, i.e,

$$Z = \{\Lambda \in \mathbf{G} : \Lambda \cap X \neq \emptyset\}.$$

1. Show that Z is a closed subvariety of $\mathbf{G}$.
2. What is the dimension of Z?
3. Show that Z is irreducible if and only if X is.

3c. Show that the complex projective plane $\mathbf{P}^2_{\mathbf{C}}$ does not nontrivially cover any other manifold.

4c. Let X be the figure eight:

8

How many connected, 3-sheeted, covering spaces of X are there, up to isomorphism over X? Prove your answer.

5c. Let $f(z)$ be regular on the closed disk $|z| \leq 1$ and not equal zero on the boundary. Show that the maximum of $\operatorname{Re} zf'(z)/f(z)$ on the boundary is at least the number of zeros of f in the disk.

6c. We recall some basic definitions from measure theory. Suppose that F is a subset of $\mathbf{R}^n$ and s is a non-negative number. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}$$

where the *diameter* $|U|$ of a set U is the greatest distance between any two points in U, and a δ*-cover* is a countable cover of F by sets of diameter at most δ. The s*-dimensional Hausdorff measure* of a set F is the limit

$$\mathcal{H}^s = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

and the *Hausdorff dimension* of a set F is the infimum of the s for which $\mathcal{H}^s(F) = 0$ (equivalently, the supremum of the s for which $\mathcal{H}^s(F) = \infty$).

The *middle third Cantor set* C is the subset of the unit interval $[0, 1]$ consisting of points that have a base 3 expansion not containing the digit 1. What is the Hausdorff dimension of the set C?

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday October 2, 2001 (Day 1)

Each question is worth 10 points, and parts of questions are of equal weight.

1a. Let X be a measure space with measure μ. Let $f \in L^1(X,\mu)$. Prove that for each $\epsilon > 0$ there exists $\delta > 0$ such that if A is a measurable set with $\mu(A) < \delta$, then
$$\int_A |f| d\mu < \epsilon.$$

2a. Let P be a point of an algebraic curve C of genus g. Prove that any divisor D with $\deg D = 0$ is equivalent to a divisor of the form $E - gP$, where $E > 0$.

3a. Let f be a function that is analytic on the annulus $1 \leq |z| \leq 2$ and assume that $|f(z)|$ is constant on each circle of the boundary of the annulus. Show that f can be meromorphically continued to $\mathbb{C} - \{0\}$.

4a. Prove that the rings $\mathbb{C}[x,y]/(x^2 - y^m)$, $m = 1,2,3,4$, are all non-isomorphic.

5a. Show that the ellipsoid $x^2+2y^2+3z^2 = 1$ is not isometric to any sphere $x^2 + y^2 + z^2 = r$.

6a. For each of the properties P_1 through P_4 listed below either show the existence of a CW complex X with those properties or else show that there doesn't exist such a CW complex.

P1. The fundamental group of X is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$.

P2. The cohomology ring $H^*(X,\mathbb{Z})$ is isomorphic to the graded ring freely generated by one element in degree 2.

P3. The CW complex X is "finite" (i.e., is built out of a finite number of cells) and the cohomology ring of its universal covering space is not finitely generated.

P4. The cohomology ring $H^*(X,\mathbb{Z})$ is generated by its elements of degree 1 and has nontrivial elements of degree 100.

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Wednesday October 3, 2001 (Day 2)

Each question is worth 10 points, and parts of questions are of equal weight.

1b. Prove that a general surface of degree 4 in $\mathbb{P}^3_{\mathbb{C}}$ contains no lines.

2b. Let R be a ring. We say that Fermat's last theorem is false in R if there exists $x, y, z \in R$ and $n \in \mathbb{Z}$ with $n \geq 3$ such that $x^n + y^n = z^n$ and $xyz \neq 0$. For which prime numbers p is Fermat's last theorem false in the residue class ring $\mathbb{Z}/p\mathbb{Z}$?

3b. Compute the integral

$$\int_0^\infty \frac{\cos(x)}{1+x^2}\, dx.$$

4b. Let $R = \mathbb{Z}[x]/(f)$, where $f = x^4 - x^3 + x^2 - 2x + 4$. Let $I = 3R$ be the principal ideal of R generated by 3. Find all prime ideals $\wp$ of R that contain I. (Give generators for each $\wp$.)

5b. Let $\mathfrak{S}_4$ be the symmetric group on four letters. Give the character table of $\mathfrak{S}_4$, and explain how you computed it.

6b. Let $X \subset \mathbb{R}^2$ and let $f : X \to \mathbb{R}^2$ be distance non-increasing. Show that f extends to a distance non-increasing map $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\hat{f}|_X = f$. Does your construction of $\hat{f}$ necessarily use the Axiom of Choice?

(Hint: Imagine that X consists of 3 points. How would you extend f to $X \cup \{p\}$ for any 4th point p?)

QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Thursday October 4, 2001 (Day 3)

Each question is worth 10 points, and parts of questions are of equal weight.

1c. Let $S \subset \mathbb{P}^3_{\mathbb{C}}$ be the surface efined by the equation $XY - ZW = 0$. Find two skew lines on S. Prove that S is nonsingular, birationally equivalent to $\mathbb{P}^2_{\mathbb{C}}$, but not isomorphic to $\mathbb{P}^2_{\mathbb{C}}$.

2c. Let $f \in \mathbb{C}[z]$ be a degree n polynomial and for any positive real number R, let $M(R) = \max_{|z|=R} |f(z)|$. Show that if $R_2 > R_1 > 0$, then

$$\frac{M(R_2)}{R_2^n} \leq \frac{M(R_1)}{R_1^n},$$

with equality being possible only if $f(z) = Cz^n$, for some constant C.

3c. Describe, as a direct sum of cyclic groups, the cokernel of $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by left multiplication by the matrix

$$\begin{bmatrix} 3 & 5 & 21 \\ 3 & 10 & 14 \\ -24 & -65 & -126 \end{bmatrix}.$$

4c. Let X and Y be compact orientable 2-manifolds of genus g and h, respectively, and let $f : X \to Y$ be any continuous map. Assuming that the degree of f is nonzero (that is, the induced map $f^* : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is nonzero), show that $g \geq h$.

5c. Use the Rouché's theorem to show that the equation $ze^{\lambda - z} = 1$, where λ is a given real number greater than 1, has exactly one root in the disk $|z| < 1$. Show that this root is real.

6c. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function such that for all x and $y \neq 0$,

$$\frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|} \leq B,$$

for some finite constant B. Prove that for all $x \neq y$,

$$|f(x) - f(y)| \leq M \cdot |x - y| \cdot \left(1 + \log^+\left(\frac{1}{|x-y|}\right)\right),$$

where M depends on B and $\|f\|_\infty$, and $\log^+(x) = \max(0, \log x)$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

October 3, 2000 (Day 1)

Each question is worth 10 points, and parts of questions are of equal weight unless otherwise indicated.

1a. Classify (up to isomorphism) all groups of order 35.

2a. Exhibit a real valued C^∞ function on the real line that is not real analytic.

3a. Consider the curve C in $\mathbb{P}^3$ which is the image of the map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$[x, y] \mapsto [x^3, x^2y, xy^2, y^3].$$

(i) Write down 3 quadric equations giving C.

(ii) Show that if L is a line that is either tangent to C or that meets C in two points, then $C \cup L$ is cut out by two quadric equations in the space spanned by the 3 equations given in (i).

4a.

(i) Construct a nowhere dense subset of the unit interval $[0, 1]$ with positive Lebesgue measure.

(ii) Construct a subset of the unit interval $[0, 1]$ which is a countable union of nowhere dense sets and has Lebesgue measure 1.

5a. Let $X \subset \mathbb{R}^3$ be the complement of the three coordinate axes. That is, a point $(x_1, x_2, x_3) \in \mathbb{R}^3$ belongs to X if and only if at most one of the three coordinates x_1, x_2, or x_3 is 0.

(i) Is X of the same homotopy type as a finite graph? (A *graph* is a simplicial complex made up of only vertices and edges.) If not, prove it; if so, exhibit such a graph and the homotopy equivalence.

(ii) Compute $H_n(X, \mathbb{Z})$ for all integers n, where $H_n(*)$ denotes singular homology.

6a. Are the following spaces complete? Justify your answer.

(i) (*4 points*) The space of all bounded holomorphic functions in the open unit disk with sup-norm.

(ii) (*4 points*) The space of all injective bounded holomorphic functions in the open unit disk with sup-norm.

(iii) (*2 points*) The space of all injective bounded holomorphic functions f in the open unit disk such that $f'(0) = 1$, with sup-norm.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

October 4, 2000 (Day 2)

Each question is worth 10 points.

1b. Let K/F be a Galois extension of fields with Galois group S_n (the symmetric group on n letters). Prove that K is the splitting field over F of a polynomial $f(x) \in F[x]$ of degree n.

2b. Let f be a surjective map between smooth projective curves over $\mathbb{C}$.

(i) If f has degree 2, show that f must have an even number of branch points.

(ii) Show that the degree 2 map $F : \mathbb{P}^1_{\overline{\mathbb{F}}_2} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{F}}_2}$ in characteristic 2 given by $y \mapsto y^2 - y$ has only 1 branch point with ramification degree 2.

3b.

(i) Show that if f is a continuous 2π-periodic function with Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

then

$$\lim_{n \longrightarrow \infty} \hat{f}(n) = 0.$$

(ii) Show that the same is true if f is only integrable on $[-\pi, \pi]$.

4b. Let f be analytic and bounded by 1 in absolute value in the open unit disk D.

(i) Show that if $f(\alpha) = 0$ ($\alpha \in D$) then

$$|f(z)| \leq \left| \frac{z - \alpha}{1 - \overline{\alpha} z} \right|$$

for $z \in D$.

(ii) Show that if $f(\alpha_1) = f(\alpha_2) = 0$ for $\alpha_1, \alpha_2 \in D$ with $\alpha_1 \neq \alpha_2$, then

$$|f(0)| \leq |\alpha_1 \alpha_2|.$$

5b.

(i) Define the degree of a smooth map $\phi : M \longrightarrow N$, where both M and N are connected compact oriented k-dimensional differentiable manifolds without boundary.

(ii) Compute the degree of the self map $x \mapsto x^q$ defined on the group $U(2)$, (that is, the set of two-by-two complex invertible matrices A with $A^{-1} = \overline{A}^t$), where q is an arbitrary integer.

6b. Let X be a compact orientable 2-manifold without boundary. Show that two continuous maps from X to S^2 are homotopic if and only if they induce the same homomorphism on two-dimensional singular homology:

$$H_2(X, \mathbb{Z}) \to H_2(S^2, \mathbb{Z}).$$

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

October 5, 2000 (Day 3)

Each question is worth 10 points.

1c. Compute

$$\int_0^\infty \frac{dx}{1+x^6}.$$

2c. Consider the ring $R = \mathbb{C}[x, y, z]$.

(i) Describe all maximal ideals of R.

(ii) Describe all minimal nonzero prime ideals of R.

(iii) Give an example of a prime ideal in R which is neither minimal nonzero nor maximal.

3c. Find a non-zero regular differential on the projective curve $y^2z = x^3 + xz^2 + z^3$.

4c. Find a finite CW-complex X whose universal covering space $\widetilde{X}$ has the property that $H^2(\widetilde{X}, \mathbb{Z})$ is not a finitely generated group. Here $H^2(*)$ denotes singular cohomology.

(**Hint:** Use as few cells as possible.)

5c. . Consider S^2 endowed with the standard round metric. Find a continuously differentiable map

$$L : S^1 \times S^1 \longrightarrow S^2$$

such that if $h(z) \in S^1 (= SO(2))$ is the holonomy of the Levi-Civita (Riemannian) connection along the loop $w \mapsto L(z, w)$ in S^2, then $h : S^1 \longrightarrow S^1$ has non-zero degree.

6c. Let A be a linear operator on an infinite dimensional separable Hilbert space such that A maps bounded sets to subsets of compact sets (that is, A is a compact operator).

(i) Find such an A that has no eigenvectors.

(ii) Show that if A is self-adjoint, then A has an eigenvector. (**Hint:** Consider a vector x with unit norm such that $(Ax, x) = \sup_{\|y\|=1}(Ay, y)$.)

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Feb 8, 2000 (Day 1)

1a. Let $\{a_1, a_2, \dots\}$ be a sequence of real numbers.

i) Define what it means to say that $\sum a_n$ is absolutely convergent, and prove that if $\sum a_n$ is absolutely convergent and equal to S and σ is a permutation of $\mathbb{Z}^+$, then $\sum a_{\sigma(n)}$ is absolutely convergent and equal to S.

ii) Suppose that $\sum a_n$ is convergent, but *not* absolutely convergent. Prove the Riemann rearrangement theorem: for any real number s, there exists a permutation σ of $\mathbb{Z}^+$ such that $\sum a_{\sigma(n)}$ converges to s, and there also exist permutations $\sigma_\pm$ such that $\sum a_{\sigma_\pm(n)}$ diverges to $\pm\infty$

2a. Let $X \subseteq \mathbb{P}^n$ be a projective variety over an algebraically closed field k. Denote by $\mathbb{P}^{n*}$ the so-called dual projective space, parameterizing hyperplanes $H \simeq \mathbb{P}^{n-1}$ in $\mathbb{P}^n$. Consider the *universal hyperplane section*

$$\Gamma_X = \{(p, H) \in X \times \mathbb{P}^{n*} \,|\, p \in H\}.$$

i) Show that Γ_X is a closed subset of the product $X \times \mathbb{P}^{n*}$.

ii) Show that Γ_X is irreducible if and only if X is irreducible.

iii) Assuming that X is irreducible, compute the dimension of Γ_X in terms of the dimension of X.

3a. Let (X, d) be a compact metric space and suppose $f : X \to X$ is a map such that for every $x \neq y$ in X, $d(f(x), f(y)) < d(x, y)$. Show that there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.

4a. Let u be a harmonic function on the punctured disc $0 < |z| < \rho$, and assume

$$\lim_{z \to 0} zu(z) = 0.$$

Prove that there exists $\alpha \in \mathbb{C}$ and a harmonic function u_0 on the disc $|z| < \rho$ such that

$$u(z) = \alpha \log(z) + u_0(z)$$

for $0 < |z| < \rho$.

5a. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. Let $f : S^n \to S^n$ be a continuous map without fixed points. Prove that f is homotopic to the antipodal map $a(\vec{x}) = -\vec{x}$.

6a. Let k be a finite field, and $\overline{k}$ an algebraic closure.

i) For each $n \geq 1$, show that there is a unique extension k_n of degree n over k inside of $\overline{k}$.

ii) Show that k_n/k is a Galois extension, with cyclic Galois group.

iii) Show that the norm map $k_n^\times \to k^\times$ sending $x \in k_n^\times$ to the product of its Galois conjugates (over k) induces a *surjective* homomorphism from $k_n^\times$ onto $k^\times$.

iv) Determine for which pairs of positive integers (n, m) we have $k_n \subseteq k_m$ inside of $\overline{k}$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Feb 9, 2000 (Day 2)

1b. Let A be an integral domain. An A-module M is said to be *torsion-free* if for $f \in A$, $m \in M$, the condition $fm = 0$ forces $f = 0$ or $m = 0$.

i) If A is a principal ideal domain and M, N are torsion-free A-modules, then prove that $M \otimes_A N$ is torsion-free.

ii) Give an example, with proof, of a domain A and torsion-free A-modules M, N such that $M \otimes_A N$ is not torsion-free.

2b. Let k be a field and $F \in k[u, v]$ a non-constant irreducible polynomial with degree $d > 0$. Let $P = z^d F(x/z, y/z) \in k[x, y, z]$ be the associated non-constant homogenous polynomial, which we assume defines a smooth curve C in $\mathbb{P}^2_k$.

i) In terms of maps between curves, give a geometric interpretation of the statement that the equation $F(U, V) = 0$ has a solution (u, v) in the rational function field $k(t)$ with $u, v \notin k$.

ii) Using facts about the genus of connected smooth projective curves over an algebraically closed field, prove that if $n > 2$, the equation $U^n + V^n = 1$ does not have a solution in $k(t)$ with $U, V \notin k$, provided the characteristic of k does not divide n.

iii) (extra credit) Prove an analogue of (ii) with the field $k(t)$ replaced by $\mathbb{Q}$.

3b Let $\|f\| = (\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta)^{1/2}$ be a norm on the space $V = \mathbb{C}[z]$ of polynomials.

i) Show that the completion H of V with respect to this norm is a Hilbert space. Find an orthonormal basis.

ii) Compute the norm (with respect to $\|\cdot\|$) of the functional $V \to \mathbb{C}$ given by $f \mapsto f(1/3)$.

iii) For any $0 < r \leq 1$, define the norm $\|\cdot\|_r$ on V by

$$\|f\|_r = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\right)^{1/2}.$$

Let V_r denote the space V equipped with this norm. Prove that the identity map $V_1 \to V_r$ is a compact operator for all $r < 1$.

4b. Show that the function defined by

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

is analytic in the open disc $|z| < 1$, but has no analytic continuation to any larger domain.

5b. The torus $S^1 \times S^1$ is embedded in $\mathbb{R}^3$ as a surface of revolution given by

$$(\theta, \phi) \mapsto \left(\frac{1}{2}\cos\theta(3+\sin\phi), \frac{1}{2}\sin\theta(3+\sin\phi), \cos\phi\right)$$

(for $\theta, \phi \in S^1$).

i) Compute the Gaussian curvature function for the induced metric at the points (2,0,0) and (1,0,0).

ii) Find an embedding $S^1 \times S^1 \hookrightarrow \mathbb{R}^4$ for which the Gaussian curvature function of the induced metric vanishes identically.

6b. Let X be a path-connected space, and $x_0, x_1 \in X$ two points.

i) Construct an isomorphism $\pi_1(X; x_0) \simeq \pi_1(X; x_1)$, natural up to conjugation.

ii) Define what it means to say that a continuous map $Y \to X$ is a covering space, and assuming that X has a basis of opens which are path connected and simply connected, describe carefully how covering spaces of X are classified by the group $\pi_1(X; x_0)$.

iii) Construct a space which has S^2 as a double cover.

iv) Construct a non-orientable space which has the torus $S^1 \times S^1$ as a double cover.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Feb 10, 2000 (Day 3)

1c. Let $\theta = xdy - ydx + dz$, a smooth 1-form on $\mathbb{R}^3$. For each $p \in \mathbb{R}^3$, let $E_p \subseteq T_p(\mathbb{R}^3)$ denote the 2-dimensional subspace annihilated by $\theta(p)$.

i) Show that the E_p's are the fibers of a subbundle of the tangent bundle of $\mathbb{R}^3$.

ii) Prove that if X and Y are vector fields spanning E over an open set $U \subseteq \mathbb{R}^3$, then $[X, Y]$ cannot be contained in E over all of U.

iii) Find a piecewise smooth curve going from $(0,0,0)$ to $(1,0,0)$ with tangent field lying in E.

2c.

i) Define the local intersection number (at an intersection point) of two distinct irreducible closed curves in the complex projective plane.

ii) Compute the intersection number of $x^2y - z^3 = 0$ and $x^3 - xyz = 0$ at the point $[0,1,0]$.

iii) State Bezout's theorem for curves in the projective plane.

3c. Compute

$$\int_0^\infty \frac{\log(x)}{x^2 + b^2} dx$$

for $b > 0$.

4c. Let $\Sigma \subseteq [0,1]$ be a Lebesgue-measureable set. Define the *density function* f_Σ by

$$f_\Sigma(t) = \overline{\lim} \frac{1}{2\epsilon} \mu(\Sigma \cap [t-\epsilon, t+\epsilon])$$

(the limsup taken as $\epsilon \to 0$).

i) Show that almost everywhere, $f_\Sigma(t) \in \{0,1\}$.

ii) Show that for all $x \in [0,1]$,

$$\int_0^x f(t)dt = \mu(\Sigma \cap [0,x]).$$

5c. Let $f = x^4 - 5 \in \mathbb{Q}[x]$.

i) Show that f is irreducible, and calculate the degree of its splitting field K over $\mathbb{Q}$, as well as the Galois group G of K over $\mathbb{Q}$.

ii) Give an explicit injective map of groups from G into S_4, the symmetric group on 4 letters.

iii) How many subfields of K have degree 4 over $\mathbb{Q}$?

6c. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. Let the group μ_p of pth roots of unity in $\mathbb{C}$ (for a prime p) act on S^3 by $\rho(\zeta) : (z_1, z_2) \mapsto (\zeta z_1, \zeta^m z_2)$, where $m \in (\mathbb{Z}/p)^\times$ is fixed. Let M denote the quotient of S^3 by this action.

i) Compute $\pi_1(M)$.

ii) Compute $H_i(M; \mathbb{Z})$ for $1 \leq i \leq 3$.

iii) Compute $H^i(M; \mathbb{Z})$ for $1 \leq i \leq 3$.

iv) How do the previous two parts change when $\mathbb{Z}$ is replaced by $\mathbb{Q}$?

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Oct 5, 1999 (Day 1)

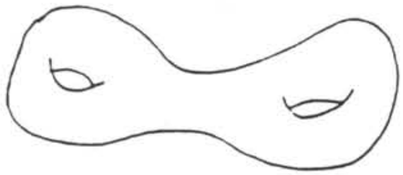
1a. Using only freshman calculus, prove that

$$\int_0^\infty \sin(t^2)\, dt = \lim_{N \longrightarrow \infty} \int_0^N \sin(t^2)\, dt$$

converges.

2a.

i) Let $f : M \longrightarrow N$ be a C^∞-map between compact, oriented, connected manifolds. Define the *degree* of f. Does there exist a degree 2 map from S^2 to a surface

$\Sigma =$ 

in $\mathbb{R}^3$ of genus 2?

ii) Consider the map $\Sigma \longrightarrow S^2$ given by assigning to $p \in \Sigma$, the unit normal X_p to Σ at p, and translating to the origin. What is its degree? Explain your answer.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

3a. Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be finite-dimensional complete normed vector spaces over $\mathbb{R}$.

i) Show that V_1 and V_2 are isomorphic as topological vector spaces.

ii) Give an example (with proof) where V_1 and V_2 are not isometric as Banach spaces.

4a. For $a \in \mathbb{R}$, show that

$$\int_0^\infty \frac{\sin(ax)}{\sinh(x)} dx = \frac{\pi}{2} \tanh \frac{a\pi}{2},$$

where $\sinh(t) = \dfrac{e^t - e^{-t}}{2} = -i \sin(it)$, $\tanh(t) = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$.

5a.

i) Let M be a smooth compact connected manifold of dimension m. What can you say about $H^m(M; \mathbb{Z}/2\mathbb{Z})$? About $H^m(M; \mathbb{Z})$? Justify your answer.

ii) Determine (with proof) the cohomology ring with coefficients in $\mathbb{Z}$ of the projective spaces $\mathbb{CP}^n$ and $\mathbb{RP}^n$.

iii) An inclusion of $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ as a hyperplane induces an inclusion

$$i : \mathbb{RP}^{n-1} \longrightarrow \mathbb{RP}^n.$$

Show that i admits no retraction (*i.e.*, there is no map $\mathbb{RP}^n \longrightarrow \mathbb{RP}^{n-1}$ with $f \cdot i = id.$)

iv) Is the same true for $i : \mathbb{RP}^n \longrightarrow \mathbb{CP}^n$, where this map is induced from the inclusion of $\mathbb{R}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ as the subset of real points?

6a.

i) Show that if A is a Noetherian commutative ring, then so is the formal power series ring $A[\![t]\!]$.

ii) Let $R = k[\![t_1, \dots, t_n]\!]$ be the ring of formal power series over a field k in the indeterminates $t_1, \dots, t_n$. Let α be a surjective endomorphism of the ring R. Show that α is an isomorphism.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Oct 6, 1999 (Day 2)

1b. Let k' be a field and G a finite group. Let V' be a non-zero, finitely generated, semisimple $k'[G]$-module. The *Brauer-Nesbitt Theorem* asserts that V' is determined up to (non-canonical) isomorphism by the characteristic polynomials $\chi_{V',g}(t) \in k'[t]$ of the actions on V' of all $g \in G$. Let $k \subseteq k'$ be a subfield over which k' is finite Galois.

i) Prove that $k' \otimes_k (\cdot)$ takes semisimple $k[G]$-modules to semisimple $k'[G]$-modules.

ii) Assume $H^2(\mathrm{Gal}(k'/k), k'^{\times}) = 1$. If $\chi_{V',g}(t) \in k[t]$ for all $g \in G$, prove that there is a $k'[G]$-module isomorphism $V' \simeq k' \otimes_k V$ for a semi-simple $k[G]$-module V which is unique up to isomorphism.

iii) Prove that $H^2(\mathrm{Gal}(k'/k), k'^{\times}) = 1$ for k'/k any extension of finite fields.

2b. Let f, g be two meromorphic functions on a compact, complex Riemann surface. Show that there exists a polynomial $F(X, Y) \neq 0$ such that $F(f, g) = 0$.

3b.

i) What is meant by an eigenfunction of a differential equation with respect to specified boundary conditions?

ii) Find a basis of eigenfunctions for the differential equation

$$u_{xx} + u_{yy} + \lambda u = 0$$

on $[0,1] \times [0,1]$ for the boundary conditions:

$$u(x,0) = \frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial y}(x,1) = u(1,y) = 0.$$

iii) What are the eigenvalues λ that arise in (ii)?

4b. Let $f = \sum a_n z^n$ be a function that is holomorphic on $\Delta = \{|z| < 1\}$ and meromorphic on some neighborhood of $\overline{\Delta} = \{|z| \leq 1\}$.

i) Assume that f has no poles on $\partial\overline{\Delta}$. Show that $|a_n| \longrightarrow 0$ as $n \longrightarrow \infty$.

ii) Assume that f has at worst simple poles on $\partial\overline{\Delta}$. Show that $\sup |a_n| < \infty$.

iii) Give a counterexample to (ii) if one allows poles on $\partial\overline{\Delta}$ with order greater than 1.

5b. Let M be a smooth manifold, and let $f : M \longrightarrow \mathbb{R}$ be a C^∞ function with no critical values in $[0,1]$. Show that $f^{-1}(0)$ and $f^{-1}(1)$ are diffeomorphic.

6b. Let X be the figure-eight: $X = \infty$.

i) How many connected, 3-sheeted covering spaces of X are there, up to isomorphism, over X? Draw them.

ii) How many of these are normal (*i.e.* Galois) covering spaces?

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Oct 7, 1999 (Day 3)

1c.

i) Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function. In what sense is the differential equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$$

equivalent to the variational condition:

$$\delta \int F(x, y, y')dx = 0?$$

ii) Prove this equivalence.

iii) Show that for the differential of length

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

in the upper half-plane $y > 0$, the geodesics (shortest paths) are the curves

$$(x-a)^2 + y^2 = r^2, \quad y > 0.$$

2c. Let k be a field, and let $f_1, \ldots f_m \in k[x_1, \ldots, x_n]$ be polynomials.

i) State the Nullstellensatz in $K[x_1, \ldots, x_n]$ for K an algebraically closed field.

ii) Use it to prove that if the f_i's have no common zero in an algebraic closure $\overline{k}$ of k, then $(f_1, \ldots, f_m) = (1)$ as ideals in $k[x_1, \ldots, x_n]$.

iii) If $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$ have a common zero in $\overline{\mathbb{F}}_p$ for infinitely many p, prove that they have a common zero in $\overline{\mathbb{Q}}$.

iv) Give a counterexample to (iii) if we replace $\overline{\mathbb{F}}_p$, $\overline{\mathbb{Q}}$ by $\mathbb{F}_p$, $\mathbb{Q}$ respectively.

3c.

i) State the Schwarz Lemma.

ii) Let $\phi : D \longrightarrow \mathbb{C}$ be a conformal mapping of the open unit disc D, and let $f : D \longrightarrow \mathbb{C}$ be a holomorphic map with $f(D) \subseteq \phi(D)$, $f(0) = \phi(0)$. Prove that $|\phi^{-1}(f(z))| \leq |z|$ for all $z \in D$.

iii) Let $f : D \longrightarrow \mathbb{C}$ be a holomorphic function, with $f(0) = 0$ and $|Re(f(z))| \leq 1$. Show that

$$|Re(f(z))| \leq \frac{4}{\pi} \tan^{-1}(|z|)$$

Prove that the equality is possible only for

$$f(z) = \frac{2}{i\pi} \log \frac{1 + e^{i\alpha z}}{1 - e^{i\alpha z}}, \qquad \alpha \in \mathbb{R}$$

4c. For $f \in L^1(\mathbb{R})$, prove that

$$\lim_{h \longrightarrow 0} \frac{1}{h} \int_{|y-x|<h} |f(y) - f(x)| dy = 0$$

for almost every x.

5c. Let k be a field, and let $f \in k[x]$ be a monic polynomial.

i) Describe in detail what it means to say that a finite extension L/k is a splitting field of f, and prove the existence and uniqueness (up to non-canonical isomorphism) of such an extension.

ii) Suppose that $f(x) = x^n - 1$, with $char(k) \nmid n$. Let L/k be a splitting field of f, so that L/k is Galois. Construct a *canonical* injection of groups $Gal(L/k) \hookrightarrow (\mathbb{Z}/n)^{\times}$. When $n = p^r$ is a prime power, determine(with proof) the image of this map for $k = \mathbb{Q}$, $\mathbb{R}$ and $\mathbb{F}_l$ where l is prime, $l \neq p$.

iii) Prove the existence of a Galois extension $L/\mathbb{Q}$ with $Gal(L/\mathbb{Q})$ cyclic of order 2^{1999}, and prove that for any such L, the unique quadratic subfield $\mathbb{Q}(\sqrt{d}) \subseteq L$ must have $d > 0$.

6c. Let $f : X \longrightarrow Y$ be a surjective map between projective smooth algebraic curves over $\mathbb{C}$.

i) State the Hurwitz genus formula for f.

ii) If f has degree 2, show that f must have an even number of branch points on Y.

iii) Show that the (degree 2) map $F : \mathbb{P}^1_{\overline{\mathbb{F}}_2} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{F}}_2}$ in characteristic 2 given by $y \mapsto y^2 - y$ has only 1 branch point, with ramification degree 2.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics

Wednesday, February 10, 1999
(Day 1)

1. **a:** Let X be a Banach space. Define the *weak topology* on X by describing a basis for the topology.
   b: Let $A : X \to Y$ be a linear operator between Banach spaces that is continuous from the weak topology on X to the norm topology on Y. Show that $A(X)$ is finite-dimensional.

2. Prove there is an entire function $f(z)$ whose zero set consists of the positive integers.

3. **a:** Compute the fundamental group of a figure 8 (see below). No justification is required.

FIGURE 8

   b: Show that any finitely-generated subgroup of the free group $\mathbb{Z} * \mathbb{Z}$ is also free.

4. Let R be the ring of continuous functions on the unit interval $[0, 1]$. Construct (with proof) an ideal in R which is not finitely generated.

5. **a:** Consider the ellipse

$$x^2 + 2y^2 = 1.$$

   Is it isometric (as a Riemannian manifold) to some circle?
   b: Consider the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 1.$$

   Is it isometric (as a Riemannian manifold) to some sphere?
   Prove that your answers are correct.

6. Consider the surface $S = \{[x : y : z : w] \in \mathbb{CP}^3 \mid xy - zw = 0\}$ in complex projective 3-space.

   a: Prove that S is non-singular.
   b: Prove that S is birationally equivalent to the projective plane.
   c: Find two lines on S that do not intersect.
   d: Prove that S is not isomorphic with the projective plane.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics

Thursday, February 11, 1999
(Day 2)

1. Show that there exists a finite extension K of $\mathbb{Q}$ with $[K:\mathbb{Q}] = 9$ such that the extension $K/\mathbb{Q}$ is only ramified over the prime 1999.

2. Let G be the Lie group of all isometries of the Euclidean plane. Introduce a coordinate system on G. Describe a left Haar measure on G in these coordinates. Show that this measure is bi-invariant.

3. **a:** Let R be an irreducible polynomial in two complex variables such that the set $\{(x,y) \in \mathbb{C}^2 : R(x,y) = 0\}$ is a non-singular curve. Consider the form $\frac{dx}{R_y}$ on the open set of the curve where $R_y = \frac{\partial R}{\partial y}$ is nowhere zero. Show that this differential form has a unique extension to a holomorphic form on the whole affine curve.
   b: Let $x^5+y^5+z^5 = 0$ define a curve Σ in $\mathbb{P}^2$. Write down an explicit basis for the holomorphic 1-forms on Σ.

4. Let $f(z)$ be an entire function with no zeros, such that $|f(z)| \le Ce^{|z|}$ for some real constant C. Show that $f(z) = \exp(az+b)$ for some $a, b \in \mathbb{C}$.

5. Let $\mathrm{O}_n(\mathbb{R})$ denote the group of n by n orthogonal matrices over $\mathbb{R}$. Compute $\pi_2(\mathrm{O}_n(\mathbb{R}))$ for all $n = 1, 2, 3, \ldots$.

6. Let a and b be real numbers.
   a: Compute the convolution (on $\mathbb{R}$) of the functions e^{-ax^2} and e^{-bx^2}.
   b: Compute a formula for the moments

$$\int_{\mathbb{R}} x^n e^{-ax^2}\,dx,$$

   where n is a non-negative integer.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics

Friday, February 12. 1999
(Day 3)

1. Evaluate the definite improper integral

$$\int_0^\infty \frac{\cos(ax)}{\cosh(bx)}\,dx,$$

where $b > 0$ and a is an arbitrary real number. (Recall that $\cosh(t) = \cos(it) = \frac{1}{2}(e^t + e^{-t})$ is the hyperbolic cosine of t.)

2. Show that a finitely generated group G has only a finite number of subgroups of a given index d.

3. Give a necessary and sufficient condition on the Fourier coefficients of a continuous function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ which characterizes the real-analytic functions.

4. **a:** Let E be an elliptic curve over $\mathbb{C}$ and $\mathbf{0} \in E$ be a choice of zero for a group law on E. Show that the ring of endomorphisms of $(E, \mathbf{0})$ is commutative.

   b: Let E be the elliptic curve $x^3 + y^3 + z^3 = 0$ in $\mathbb{P}^2$ over an algebraic closure $\bar{\mathbb{F}}_5$ of $\mathbb{F}_5$. Let $[1 : -1 : 0]$ be its zero element. Show that the ring of endomorphisms of $(E, [1 : -1; 0])$ is not commutative.

5. Let K be the Klein bottle.

   a: Compute $\pi_1(K)$.

   b: Compute the homology groups $H_n(K; \mathbb{Z})$ for all $n \geq 0$. You may use that K is not orientable.

6. Consider a parametric curve $f : \mathbb{R} \to \mathbb{R}^3$, where f is a C^1-map.

   a: State an easy condition on the function f that guarantees that f is a smooth path (i.e., the range of f is a smooth curve which may cross itself but has no corners).

   b: State what is meant by the arc-length parametrization and prove that the condition you have given in a is sufficient to ensure that the arc-length parametrization of the curve f is also of class C^1.

   c: Assuming that f itself gives the arc-length parametrization, define the curvature, the torsion, the principal normal and the binormal; and state the Frenet-Serret formulas.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Monday, September 28, 1998 (Day 1)

1a. i) Define *Noetherian ring.*
ii) Prove Hilbert's Basis Theorem: If A is a Noetherian ring, then $A[x]$ is Noetherian.
iii) Give, with justification, an example of a non-Noetherian ring.

2a. Let X and Y be Banach spaces and $S, T : X \to Y$ bounded linear maps.

i) Prove that if T is a bijection then there exists $\delta > 0$, depending only on T, such that $\|S - T\| < \delta \Rightarrow S$ is a bijection.
ii) Assume that the kernel of T is finite-dimensional, the image of T is closed, and the cokernel of T is finite dimensional. Prove that there exists $\delta > 0$ depending only on T, such that if $\|S - T\| < \delta$ then all three properties (finite-dimenionsal kernel and cokernel, closed image) hold for S.

3a. Let $L \subset \mathbb{C}$ be a lattice and define

$$\wp(z) := \frac{1}{z^2} + \sum{}' \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right),$$

where $\sum'$ means the sum over all *nonzero* $l \in L$.

i) Show that $\wp$ is convergent and analytic on $\{z \in \mathbb{C} : z \notin L\}$.
ii) Show that $\wp$ is *periodic with respect to* L, i.e. $\wp(z + l) = \wp(z)$ for all $l \in L$.
iii) Show that any holomorphic function on $\mathbb{C}$ which is periodic with respect to L is constant. Using this, deduce that

$$(d\wp/dz)^2 = 4\wp^3 - g_2(L)\wp - g_3(L),$$

where

$$g_2(L) := 60 \sum{}' \frac{1}{l^4}, \quad g_3(L) := 140 \sum{}' \frac{1}{l^6}.$$

4a. Compute $\pi_n(\mathbb{CP}^k)$ for all positive integers n, k such that $2k+1 \geq n$.

5a. i) Find non-constant rational functions $x, y \in \mathbb{C}(t)$ such that $x^2+y^2=1$.
ii) Prove that for $n \geq 3$ there do not exist non-constant $x, y \in \mathbb{C}(t)$ such that $x^n+y^n=1$.

6a. Define a metric on $\{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$ by the line element

$$ds^2 = \frac{dr^2 + r^2 d\theta^2}{(1-r^2)^p}.$$

Here (r, θ) are polar coordinates and $p \in \mathbb{R}$.

i) Compute the connection form relative to the orthonormal frame

$$e^r = \frac{dr}{(1-r^2)^{p/2}}, \quad e^\theta = \frac{r\, d\theta}{(1-r^2)^{p/2}}.$$

ii) Compute the second fundamental form of the circle $r = 1/2$.
iii) For which p is this circle a geodesic?
iv) Compute the Gaussian curvature of this metric.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Tuesday, September 29, 1998 (Day 2)

1b. For $m > 0$ prove that

$$\int_0^\infty \frac{e^{-mx^2}\cos(mx)}{e^{\pi x}+e^{-\pi x}}dx = \frac{e^{-m/4}}{4}.$$

(Hint: The denominator of the integrand has a pole at $x = i/2$.)

2b. Let R be a nonzero commutative ring with unity, and I a proper ideal in R (i.e. an ideal other than R itself).

i) Show that the *radical* of I (the set of $r \in R$ such that $r^n \in I$ for some integer $n > 0$) is the intersection of the prime ideals of R which contain I.

ii) State the Hilbert Nullstellensatz for rings R which are finitely generated over an algebraically closed field. Explain how this theorem implies that if R is a polynomial ring over $\mathbb{C}$ in finitely many variables then the radical of I is the intersection of all *maximal* ideals of R which contain I.

iii) Give an example of a ring R and a proper ideal $I \subset R$ whose radical is not the intersection of the maximal ideals which contain it.

3b. Let G be the group $\mathrm{GL}_3(\mathbb{F}_2)$.

i) How many conjugacy classes does G have?

ii) Show that G has exactly two conjugacy classes of size 24.

4b. Let $T_n = \mathbb{R}^n/\mathbb{Z}^n$ be the torus of dimension n.

i) Describe the cohomology ring of T_n.
ii) If $\alpha : T_n \to T_n$ sends $\vec{x}$ to $3\vec{x}$, compute the "Lefschetz number" of α:

$$L(\alpha) := \sum_{i=0}^{n} (-1)^i \operatorname{Trace}(\alpha^*|_{H^i(T_n)}).$$

iii) Prove that the graph of α is transverse to the diagonal in $T_n \times T_n$.
iv) State the Lefschetz fixed point theorem and check that it holds in this case.

5b. Let S be a compact connected surface in $\mathbb{R}^3$ with Gaussian curvature everywhere positive.

i) Show that S is diffeomorphic to the sphere. (You should not use the classification of compact surfaces.)
ii) Show that S is *convex*, i.e. lies to one side of each of its tangent planes.
iii) Show that S contains at least one *umbilical point*, i.e. a point at which the principal curvatures are equal. (Hint: S^2 has no nowhere vanishing vector fields.)

6b. Consider the system of differential equations

$$\frac{dx}{dt} = ax + 4y,$$

$$\frac{dy}{dt} = -x + 2y.$$

Discuss, with appropriate sketches, the nature of the solution curves $(x(t), y(t))$ for various values of a. What are the critical values of a where the nature of the solution changes?

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Thursday, October 1, 1998 (Day 3)

1c. Two sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ of positive numbers are said to be *asymptotic* if $\lim_{n\to\infty}(x_n/y_n) = 1$. This is denoted $x_n \sim y_n$.

i) Prove that if $x_n \sim y_n$ and $\sum_{n=1}^{\infty} x_n = \infty$ then

$$\sum_{k=1}^{n} x_k \sim \sum_{k=1}^{n} y_k.$$

ii) Show that for each $a > 1$ there exist positive constants $c_1(a), c_2(a)$ such that

$$\sum_{k=1}^{n} \frac{a^k}{k} \sim c_1(a) \int_1^{n+1} \frac{a^x}{x}\, dx \sim c_2(a) \frac{a^n}{n}.$$

Compute $c_1(a)$ and $c_2(a)$ as explicit functions of a.

2c. Define a vector field X on $\mathbb{R}^2$ by

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

i) Find all points of $\mathbb{R}^2$ at which X is tangent to the curve $2x^2 - 4xy + 3y^2 = 1$.
ii) Determine the Lie derivative of $3x^3\, dx - y^2\, dy$ in the direction of the vector field X.
iii) Let M be a smooth manifold, ω a smooth differential form on M, and X a smooth vector field on M. Give Cartan's general formula relating the exterior derivative of ω to the Lie derivative of ω in the direction of X.

3c. Let $L \subset \mathbb{C}$ be a lattice; let $X = \mathbb{C}/L$, a Riemann surface of genus 1.

i) Show that any holomorphic map $f : X \to X$ is induced from a map $z \mapsto az + b$ on $\mathbb{C}$ for suitable $a, b \in \mathbb{C}$.

ii) Let τ_1, τ_2 be distinct holomorphic automorphisms of X of order 2, and let $\sigma = \tau_1\tau_2$. Assume that each of τ_1, τ_2 has fixed points on X. Prove that σ has no fixed points.

4c. i) Let C_1, C_2 be two distinct irreducible curves in $\mathbb{CP}^2$. Define the intersection multiplicity $I(P; C_1, C_2)$ of C_1, C_2 at a point $p \in C_1 \cap C_2$, and state Bezout's theorem.

ii) Using Bezout's theorem, prove that there exist finite sets of points in $\mathbb{CP}^2$ whose homogeneous ideal cannot be generated by two elements.

5c. A finite group G will be said to have "property R" if, for all $g \in G$ and every integer n relatively prime to the order of G, the group elements g and g^n are conjugate in G.

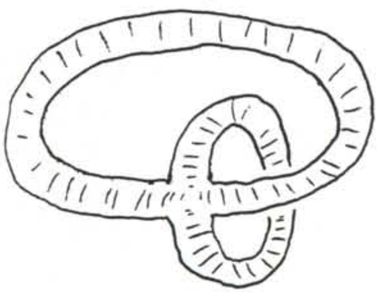
i) Give infinitely many non-isomorphic examples of finite groups having property R.

ii) If G has property R, prove that for every finite-dimensional representation

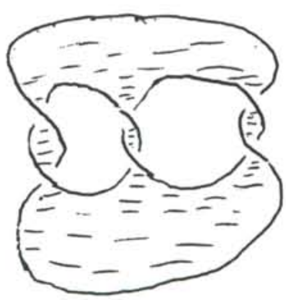
$$\rho : G \to \mathrm{GL}_m(\mathbb{C})$$

of G we have $\mathrm{Trace}(\rho(g)) \in \mathbb{Z}$ for all $g \in G$.

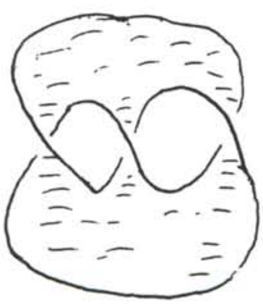
6c. Among the three compact surfaces S_1, S_2, S_3 drawn below, for which pairs $\{i, j\}$ are S_i and S_j homeomorphic? Justify your answer.



S_1



S_2



S_3

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Tuesday, February 17, 1998 (Day 1)

1. For any linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$, let $A(T)$ be the algebra of linear maps $\mathbb{C}^n \to \mathbb{C}^n$ that commute with T. Give necessary and sufficient conditions on two semisimple (i.e., diagonalizable) linear maps $S, T : \mathbb{C}^n \to \mathbb{C}^n$ for $A(S)$ to be isomorphic to $A(T)$ as algebras.

2. Let X and Y be compact connected oriented 3-manifolds, with $\pi_1(X) = \mathbb{Z}/5\mathbb{Z}$ and $\pi_1(Y) = \mathbb{Z}/10\mathbb{Z}$. Find $H_n(X \times Y, \mathbb{Z})$ for all $n \geq 0$.

3. Prove that if the closed unit ball of a Hilbert space V is compact, then V is finite-dimensional.

4. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be the *Fermat quartic*, that is, the Riemann surface

$$C = \{[X, Y, Z] : X^4 + Y^4 + Z^4 = 0\}.$$

a. What is the genus of C?

b. Let $x = X/Z$ and $y = Y/Z$. Show that the meromorphic 1-form on the open set $Z \neq 0$ of C given by

$$\omega = \frac{dx}{y^3}$$

extends to a global holomorphic 1-form on all of C, and that the forms $x\omega$ and $y\omega$ do as well. Show that $\{\omega, x\omega, y\omega\}$ is a basis for the space of global holomorphic 1-forms on C.

c. Prove that C is not hyperelliptic, that is, there does not exist a meromorphic function on C with exactly two poles.

5. Evaluate the integral

$$\int_0^\infty \frac{\cos x \, dx}{\alpha^2 + x^2} \quad \text{for} \quad \alpha > 0.$$

6. For any topological space X, we define the *symmetric square* S^2X of X to be the quotient of the product $X \times X$ by the involution exhanging factors—that is, the set of unordered pairs of points of X, given the quotient topology. For the following, let X be a compact 2-manifold.

a. Prove that S^2X is a 4-manifold. Under what conditions is it orientable?

b. Find the Euler characteristic of S^2X in terms of that of X.

c. Describe S^2X explicitly in case X is the 2-sphere.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Wednesday, February 18, 1998 (Day 2)

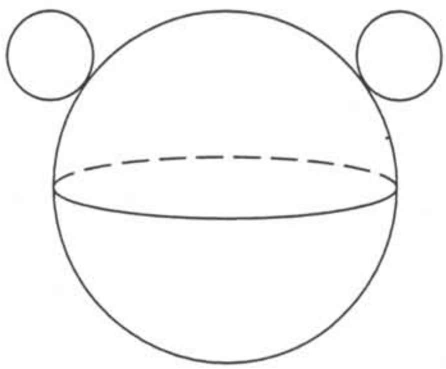
1. Describe (as a direct sum of cyclic groups) the cokernel of the map $\phi : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by left multiplication by the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

2. Let X be the space obtained from the sphere

$$S^2 = \{(x, y, z : x^2 + y^2 + z^2 = 1\}$$

by identifying $(1, 0, 0)$ with $(-1, 0, 0)$ and also identifying $(0, 1, 0)$ with $(0, -1, 0)$. Let Y be the torus $S^1 \times S^1$, and let Z be the space $S^2 \vee S^1 \vee S^1$ obtained by attaching two circles to S^1:



a. Compute the cohomology groups of these three spaces.
b. Classify them up to homotopy type.

3. Let K be the splitting field of the polynomial $x^4 - 2$ over $\mathbb{Q}$, and let G be the Galois group of K over $\mathbb{Q}$.
a. Show that G is a dihedral group.
b. Describe all subfields of K.

4. Find the Fourier series for the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = t \text{ for } -\pi < t \leq \pi$$

and

$$f(t + 2\pi) = f(t) \text{ for all } t.$$

Deduce from this the values of the sums

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

and

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

5. Let $X = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Show that every biholomorphic map $X \to X$ is given by a linear fractional transformation $z \mapsto \frac{az+b}{cz+d}$.

6. Let m and n be positive integers, and k a positive integer less than m or n. Let $N = mn - 1$, and realize the projective space $\mathbb{P}^N_{\mathbb{C}}$ as the space of nonzero $m \times n$ matrices modulo scalars. Let $X_k \subset \mathbb{P}^N_{\mathbb{C}}$ be the subset of matrices of rank k or less.

a. Show that X_k is an algebraic subvariety (that is, closed algebraic subset) of $\mathbb{P}^N_{\mathbb{C}}$.

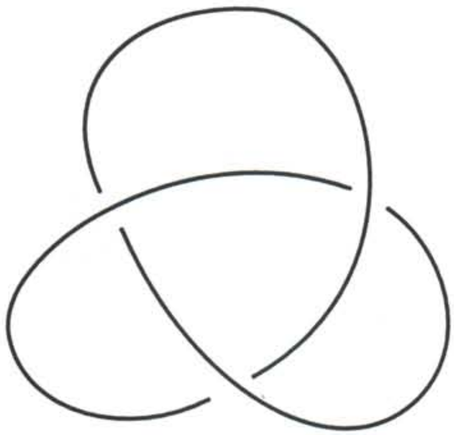
b. What is the dimension of X_k?

c. Show that X_k is irreducible.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Thursday, February 19, 1998 (Day 3)

1. Consider the ring $R = \mathbb{C}[x, y, z]$.
   a. Describe all maximal ideals in R.
   b. Describe all minimal prime ideals in R.
   c. Give an example of a prime ideal that is neither maximal or minimal.

2. Let $K \subset \mathbb{R}^3$ be the trefoil knot, as pictured:



a. Find a finite presentation of the fundamental group $\pi_1(\mathbb{R}^3 \setminus K)$ of the complement of K in $\mathbb{R}^3$.

b. Show that $\pi_1(\mathbb{R}^3 \setminus K)$ admits a presentation with two generators X and Y and one relation $XYX = YXY$.

c. Find a surjective homomorphism from $\pi_1(\mathbb{R}^3 \setminus K)$ onto the symmetric group $\mathfrak{S}_3$ on three letters.

d. Show that K is knotted.

3. Show that

$$\int_0^1 \frac{1 + x^{30}}{1 + x^{60}} dx \; = \; 1 + \frac{c}{30} \quad \text{where} \quad 0 < c < 1 \,.$$

4. Let $C \subset \mathbb{R}^3$ be the curve given by the equations $y = x^2$ and $z = xy$. Find the osculating plane to C at the point $(1, 1, 1) \in C$, and compute the curvature and torsion of C at that point.

5. Find the Laurent expansion

$$f(z) = \sum_{n \in \mathbf{Z}} a_n z^n .$$

around 0 of the function

$$f(z) = \frac{1}{z^2 + z + 1}$$

a. valid in the open unit disc $\{z : |z| < 1\}$; and
b. valid in the complement $\{z : |z| > 1\}$ of the closed unit disc in $\mathbb{C}$.

6. Let $\mathfrak{S}_4$ be the symmetric group on four letters. Give the character table of $\mathfrak{S}_4$, and explain how you computed it.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Tuesday, October 14, 1997 (Day 1)

1. Classify all groups of order 55.

2. Let X be the topological space obtained by identifying the two boundary components of $S^n \times I$ via the antipodal map—that is,

$$X = S^n \times I/(x,0) \sim (-x,1)\ \forall x \in S^n .$$

Find the homology groups of X.

3. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x+x^2}\,dx .$$

4. Let V be the vector space of all homogeneous polynomials $F(X,Y,Z)$ of degree d in three variables over $\mathbb{C}$, and let $\mathbb{P}V \cong \mathbb{P}^{\binom{d+2}{2}-1}$ be the projective space of such polynomials mod scalars.

a. Let $\Sigma \subset \mathbb{P}V$ be the set of *reducible* polynomials, that is, products of polynomials of lower degree. Why is Σ an algebraic subvariety of $\mathbb{P}V$?

b. Let $F_1, \ldots, F_d$ be any d homogeneous polynomials of degree d in three variables over $\mathbb{C}$. Show that there exists scalars $c_1, \ldots, c_d \in \mathbb{C}$, not all zero, such that $\sum c_i F_i$ is reducible.

5. a. State the open mapping theorem.

b. Let $\phi : V \to W$ be a linear map of Banach spaces. Suppose that whenever $v_i \in V$ is a sequence of vectors such that $v_i \to 0$ in V and $\phi(v_i) \to w$ for some $w \in W$, then $w = 0$. Show that ϕ is continuous.

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6. For any pair of real numbers a and b, let $M_{a,b}$ be the matrix

$$M_{a,b} = \begin{pmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{pmatrix}$$

with entries a on the diagonal and b off the diagonal.

a. What are the eigenvalues of $M_{a,b}$?

b. For what values of a and b does the limit $\lim_{n\to\infty}(M_{a,b})^n$ exist?

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Wednesday, October 15, 1997 (Day 2)

1. Let X and Y be compact orientable 2-manifolds of genus g and h respectively, and let $f : X \to Y$ be any continuous map. Assuming that the degree of f is nonzero (that is, the induced map $f_* : H_2(X, \mathbb{Z}) \to H_2(Y, \mathbb{Z})$ is nonzero), show that $g \geq h$.

2. Let $V \cong \mathbb{C}^2$ be the standard representation of the group $\mathrm{SL}(2, \mathbb{C})$, and let $V^{\otimes n}$ be the n^{th} tensor power of V. How many factors are there in the expression of $V^{\otimes 2}$ as a direct sum of irreducible representations? How about $V^{\otimes 3}$ and $V^{\otimes 4}$?

3. Define a sequence of integers c_n for $n \geq 0$ inductively by setting $c_0 = 1$ and

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}$$

for $n \geq 1$. Show that

$$c_n = \frac{(2n)!}{n!(n+1)!}.$$

4. Let L be the splitting field of the polynomial $f(x) = x^4 - 3x^2 + 1$ over $\mathbb{Q}$, that is, the field obtained by adjoining to $\mathbb{Q}$ all four roots of f. Find the Galois group of L over $\mathbb{Q}$.

Oct 15, 97
Day 2

5. Let $\Sigma \subset [0,1]$ be a Lebesgue measurable subset. Define the *density function* $f(t)$ of the subset Σ for $t \in (0,1)$ by

$$f(t) \;=\; \limsup_{\epsilon \to 0} \frac{1}{2\epsilon}\mu(\Sigma \cap [t-\epsilon, t+\epsilon])\,.$$

a. Show that $f(t) = 0$ or 1 almost everywhere.
b. Show that for any $x \in [0,1]$,

$$\int_0^x f(t)dt \;=\; \mu(\Sigma \cap [0,x])\,.$$

6. For any $z \in \mathbb{C} \setminus \mathbb{Z}$, consider the limit

$$f(z) \;=\; \lim_{N \to \infty} \left(\sum_{n=-N}^{N} \frac{1}{z+n} \right).$$

a. Show that this limit converges.
b. Show that for $z \in \mathbb{C} \setminus \mathbb{Z}$, $f(z) = \pi \cot(\pi z)$.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Thursday, October 16, 1997 (Day 3)

1. Express $(\mathbb{Z}/144\mathbb{Z})^*$ (that is, the group of units in the ring $\mathbb{Z}/144\mathbb{Z}$) as a product of cyclic groups. How many elements of order 4 are there in this group?

2. Let $X \subset \mathbb{R}^3$ be a smooth oriented surface (not necessarily compact), and let $g : X \to S^2$ be the Gauss map, assigning to each point $p \in X$ the unit normal vector to X at p. We say that a point $p \in X$ is *parabolic* if the differential $dg(p) : T_pX \to T_{g(p)}S^2$ is singular.

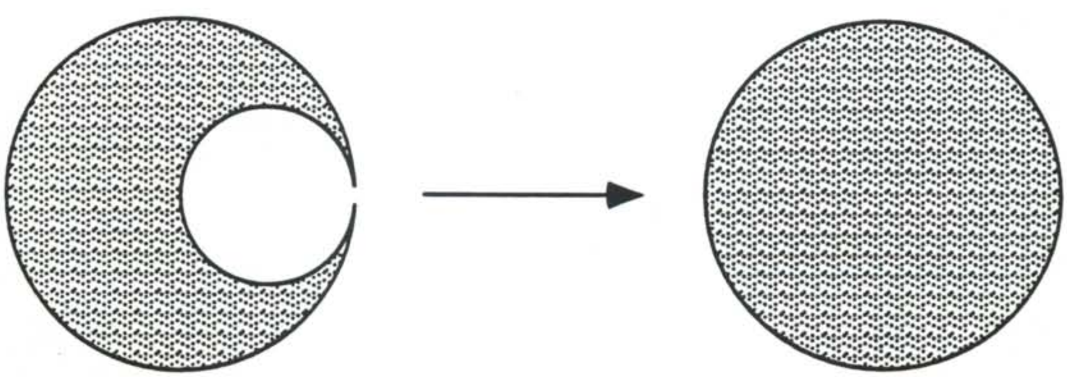
a. Find an example of a surface X such that every point of X is parabolic. Why can't such a surface be compact?

b. Suppose now that the locus of parabolic points on X is a smooth curve C, and suppose that at every point $p \in C$ the tangent line $T_pC \subset T_pX$ to C coincides with the kernel of $dg(p)$. Show that C must be planar.

3. Let $\Omega \subset \mathbb{C}$ be the open set given as

$$\Omega = \{z \in \mathbb{C} : |z| < 2 \text{ and } |z-1| > 1\}.$$

Find a conformal map of Ω onto the unit disc $\{z : |z| < 1\}$.



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4. Let $X = S^2 \vee \mathbb{RP}^2$ be the wedge of the two-sphere and the real projective plane (that is, the space $X = S^2 \cup \mathbb{RP}^2/p \sim q$ obtained from the disjoint union of S^2 and $\mathbb{RP}^2$ by identifying a point $p \in S^2$ with a point $q \in \mathbb{RP}^2$).

a. Find the homology groups of X.

b. What is the universal covering space of X?

c. Find the homotopy groups $\pi_1(X)$ and $\pi_2(X)$.

5. Let C be the space $C[0,1]$ of continuous real-valued functions on the unit interval $[0,1] \subset \mathbb{R}$, with the norm $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$. Let C^1 be the space of functions on $[0,1]$ with continuous first derivative, with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that the inclusion of C^1 in C is a compact operator.

6. Let M be a compact C^∞ manifold and $f : M \to \mathbb{R}$ a C^∞ function. For every value α of f, let

$$L_\alpha = f^{-1}(\alpha)$$

be the corresponding level set, and

$$M_\alpha = f^{-1}((-\infty, \alpha)).$$

Assume that for some pair of values $\alpha < \beta$ of f, the differential $df(p)$ is nonzero for all $p \in f^{-1}([\alpha, \beta])$. Show that

a. L_α is diffeomorphic to L_β; and

b. M_α is diffeomorphic to M_β.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Tuesday, February 25, 1997 (Day 1)

1. Factor the polynomial $x^3 - x + 1$ and find the Galois group of its splitting field if the ground field is:

$$\text{a) } \mathbf{R}, \qquad \text{b) } \mathbf{Q}, \qquad \text{c) } \mathbf{Z}/2\mathbf{Z}\,.$$

2. Let A be the $n \times n$ (real or complex) matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1/n & 1/n & 1/n & \dots & 1/n \end{pmatrix}.$$

Prove that as $k \to \infty$, A^k tends to a projection operator P onto a one-dimensional subspace. Find $\ker P$ and $\operatorname{Image} P$.

3. a. Show that there are infinitely many primes p congruent to 3 mod 4.
   b. Show that there are infinitely many primes p congruent to 1 mod 4.

4. a. Let L_1, L_2 and $L_3 \subset \mathbf{P}^3_{\mathbf{C}}$ be three pairwise skew lines. Describe the locus of lines $L \subset \mathbf{P}^3_{\mathbf{C}}$ meeting all three.
   b. Now let L_1, L_2, L_3 and $L_4 \subset \mathbf{P}^3_{\mathbf{C}}$ be four pairwise skew lines. Show that if there are three or more lines $L \subset \mathbf{P}^3_{\mathbf{C}}$ meeting all four, then there are infinitely many.

5. a. State the Poincaré duality and Kunneth theorems for homology with coefficients in $\mathbf{Z}$ (partial credit for coefficients in $\mathbf{Q}$).
   b. Find an example of a compact 4-manifold M whose first and third Betti numbers are not equal, that is, such that $H^1(M, \mathbf{Q})$ and $H^3(M, \mathbf{Q})$ do not have the same dimension.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Wednesday, February 26, 1997 (Day 2)

1. Define a metric on the unit disc $\{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$ by the line element

$$ds^2 = \frac{dr^2 + r^2 d\theta^2}{(1-r^2)^p}.$$

Here (r,θ) are polar coordinates and p is any positive real number.

a. Compute the connection form relative to the orthonormal frame

$$e^r = \frac{dr}{(1-r^2)^{p/2}}, \qquad e^\theta = \frac{r d\theta}{(1-r^2)^{p/2}}.$$

b. Compute the second fundamental form for the circle $r = 1/2$.

c. For which p is this circle a geodesic?

d. Compute the Gaussian curvature of this metric.

2. Let C be the space $C[0,1]$ with the sup norm $||f||_\infty$. Let C^1 be the space $C^1[0,1]$ with the norm $||f|| = ||f||_\infty + ||f'||_\infty$. Prove that the natural inclusion $C^1 \subset C$ is a compact operator.

3. Let X be a compact Riemann surface, and let f and g be two meromorphic functions on X. Show that there exists a polynomial $P \in \mathbb{C}[X,Y]$ such that $P(f(z), g(z)) \equiv 0$.

4. Let $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2+|w|^2 = 1\}$. Let p be a prime and m an integer relatively prime to p. Let ζ be a primitive p^{th} root of unity, and let the group $G = \mathbb{Z}/p\mathbb{Z}$ act on S^3 by $(\zeta,(z,w)) \mapsto (\zeta z, \zeta^m w)$. Let $M = S^3/G$.

a. compute $\pi_i(M)$ for $i = 1$, 2 and 3.

b. compute $H_i(M,\mathbb{Z})$ for $i = 1$, 2 and 3.

c. compute $H^i(M,\mathbb{Z})$ for $i = 1$, 2 and 3.

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Day 2

5. Let d be a square-free integer. Compute the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{d})$. Give an example where this ring is not an integral domain.

6. Prove that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Thursday, Feb. 27, 97 (Day 3)

1. Let $\alpha : (0,1) \to \mathbf{R}^3$ be any regular arc (that is, α is differentiable and α' is nowhere zero). Let $\mathbf{t}(u)$, $\mathbf{n}(u)$ and $\mathbf{b}(u)$ be the unit tangent, normal and binormal vectors to α at $\alpha(u)$. Consider the *normal tube of radius* ϵ around α, that is, the surface given parametrically by

$$\phi(u,v) \;=\; \alpha(u) + \epsilon\cos(v)\mathbf{n}(u) + \epsilon\sin(v)\mathbf{b}(u)\,.$$

a. For what values of ϵ is this an immersion?

b. Assuming that α itself has finite length, find the surface area of the normal tube of radius ϵ around α.

The answers to both questions should be expressed in terms of the curvature $\kappa(u)$ and torsion $\tau(u)$ of α.

2. Recall that a *fundamental solution* of a linear partial differential operator P on $\mathbf{R}^n$ is a distribution E on $\mathbf{R}^n$ such that $PE = \delta$ in the distribution sense, where δ is the unit Dirac measure at the origin. Find a fundamental solution E of the Laplacian on $\mathbf{R}^3$

$$\Delta \;=\; \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

that is a function of $r = |x|$ alone. Prove that your fundamental solution indeed satisfies $\Delta E = \delta$.

Hint: Use the appropriate form of Green's theorem.

3. The group of rotations of the cube in $\mathbf{R}^3$ is the symmetric group S_4 on four letters. Consider the action of this group on the set of 8 vertices of the cube, and the corresponding permutation representation of S_4 on $\mathbf{C}^8$. Describe the decomposition of this representation into irreducible representations.

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Day 3

4. Suppose a_i, $i = i, \ldots, n$ are positive real numbers with $a_1 + \ldots + a_n = 1$. Prove that for any nonnegative real numbers $\lambda_1, \ldots, \lambda_n$,

$$\sum_{i=1}^{n} a_i \lambda_i^2 \geq \left(\sum_{i=1}^{n} a_i \lambda_i \right)^2$$

with equality holding only if $\lambda_1 = \ldots = \lambda_n$.

5. a. For which natural numbers n is it the case that every continuous map from $\mathbf{P}^n_{\mathbf{C}}$ to itself has a fixed point?

b. For which n is it the case that every continuous map from $\mathbf{P}^n_{\mathbf{R}}$ to itself has a fixed point?

6. Fermat proved that the number $2^{37} - 1 = 137438953471$ was composite by finding a small prime factor p. Suppose you know that $200 < p < 300$. What is p?

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Tuesday, October 15, 1996 (Day 1)

1. Find the homotopy groups $\pi_k(\mathbf{CP}^n)$ of complex projective n-space

a) for $k \leq 2n+1$; and

b) for $k = 2n+2$.

2. Consider the Banach space $X = C[a,b]$ of continuous functions on the interval $[a,b] \subset \mathbf{R}$ with the supremum norm $||\cdot||_{\mathrm{sup}}$. Let K be a continuous function on $[a,b]\times[a,b]$, and define an integral operator $T : X \to X$ by

$$(Tf)(x) = \int_a^b K(x,y)f(y)\,dy\,.$$

Prove that T is an operator norm limit of operators of finite rank. (The operator S is of finite rank iff $\dim(\mathrm{Im}(S)) < \infty$.)

3. Are the following pairs of domains conformally equivalent? Prove your answer in each case.

a) The complex plane $\mathbf{C}$ and the disc $D = \{z \in \mathbf{C} : |z| < 1\}$.

b) The punctured complex plane $\mathbf{C}^* = \{z \in \mathbf{C} : 0 < |z|\}$ and the punctured disc $D^* = \{z \in \mathbf{C} : 0 < |z| < 1\}$.

c) The complex plane with a closed disc removed (that is, $X = \{z \in \mathbf{C} : 1 < |z|\}$) and the punctured disc $D^* = \{z \in \mathbf{C} : 0 < |z| < 1\}$.

4. Let p be a prime and let G be the group $\mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/p^2\mathbf{Z}$.

a) How many subgroups of order p does G have?

b) How many subgroups of order p^2 does G have?

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Day 1
P 2

5. The unitary group $U(n)$ is the multiplicative Lie group of $n \times n$ complex matrices A such that $A\overline{A}^t = I$ ($\overline{A}^t$ is the transpose of the complex conjugate of A, and I is the $n \times n$ identity matrix).

a) What is the dimension of $U(n)$ as a real manifold?

b) Let $X = \{A \in U(n) \mid A^2 = I\}$. How many connected components does X have?

c) What is the dimension of each connected component of X?

6. a) Define the Gaussian curvature $K(p)$ and the mean curvature $H(p)$ of a surface $S \subset \mathbf{R}^3$ at a point $p \in S$.

b) Let $t \mapsto (x(t), y(t))$ be a curve in the upper half plane parametrized by arc length (that is, such that $y > 0$ and $x'^2 + y'^2 \equiv 1$), and consider its surface of revolution, parametrized by

$$(t, \theta) \mapsto (x(t), y(t)\cos\theta, y(t)\sin\theta).$$

Find H and K of this surface.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Wednesday, October 16, 1996 (Day 2)

1. Let $X = S \setminus \{p\}$ be a two-holed torus with one point removed. How many connected covering spaces of degree 2 does X have?

2. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $p > 1$ and let $(f_n)_{n \geq 1}$ be an $\mathcal{L}^p$-bounded sequence. Assume $f_n \to 0$ in measure, that is,

$$\forall \varepsilon > 0 \quad \lim_{n \to \infty} \mu(|f_n| \geq \epsilon) = 0.$$

Show that $f_n \to 0$ in $\mathcal{L}^1$.

3. Evaluate

$$\int_0^\infty \frac{\sqrt{t}}{(1+t)^2}\, dt.$$

4. Let k be a field of characteristic $p > 0$..

a) Show that the function $F(x) = x^p$ induces an injective field homomorphism $F : k \to k$.

b) Assume that $F : k \to k$ is surjective, and let $f(x)$ be an irreducible polynomial in $k[x]$. Let E be an extension field of k, and α an element of E. Show that $(x - \alpha)^2$ does not divide $f(x)$ in $E[x]$.

c) Assume again that $F : k \to k$ is surjective, and that E is a field extension of finite degree over k. Show that $F : E \to E$ is also surjective.

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Day 2

5. Let $M \subset \mathbf{R}^N$ be an n-dimensional submanifold of N-dimensional Euclidean space, and let $\gamma : [a,b] \to M \subset \mathbf{R}^N$ be a C^∞ arc of finite length on M. We will say that γ is a *geodesic* on M if it is a critical point for the length function; that is, if for any C^∞ map $\Gamma : (-\epsilon, \epsilon) \times [a,b] \to M$ with $\Gamma(0,t) = \gamma(t)$ for all t and $\Gamma(\lambda, a) = \gamma(a)$ and $\Gamma(\lambda, b) = \gamma(b)$ for all λ, the length function

$$l(\lambda) = \text{length}\big(\gamma_\lambda = \Gamma(\lambda, t) : [a,b] \to M\big)$$

has derivative zero at $\lambda = 0$.

Assume that $\|\frac{d\gamma}{dt}\| \equiv 1$. Prove that γ is a geodesic iff

$$\frac{d^2\gamma}{dt^2}$$

is normal to M at $\gamma(t)$ for all t.

6. Define a sequence of complex numbers by

$$a_0 = 1, \qquad a_n = \frac{a_{n-1}}{1} + \frac{a_{n-2}}{2} + \cdots + \frac{a_0}{n}.$$

Evaluate the limit

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Hint: consider the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Thursday, October 17, 1996 (Day 3)

1. Let $T = D^2 \times S^1$ be a solid torus, and let $S \subset T$ be a submanifold homeomorphic to S^1 and such that the generator of $\pi_1(S)$ is n times the generator of $\pi_1(T)$. For example, in terms of coordinates (z, θ) on $T = D^2 \times S^1$, you could take the image of the arc

$$\phi \mapsto (\frac{e^{i\phi}}{2}, n\phi), \quad 0 \le \phi \le 2\pi .$$

Let $X = T \setminus S$ be the complement of S in T.

a) Find the Euler characteristic $\chi(X)$.

b) Find the homology groups $H_i(X, \mathbf{Z})$.

2. Recall that the Fourier transform of a function $\varphi(t)$ is defined to be the function

$$\hat{\varphi}(u) = \int_{\mathbf{R}} e^{-itu} \varphi(t)\, dt .$$

a) Find the Fourier transform $\hat{\varphi}$ of the function $\varphi(t) = 1/(t^2 + 1)$.

b) Let $g : \mathbf{R} \to \mathbf{R}$ be a Schwartz class function (that is, g is C^∞ and all of the derivatives of g decay at infinity faster than the reciprocal of any polynomial). Consider the differential equation

$$f(t) - f''(t) = g(t). \tag{1}$$

Find a solution to (1) of the form

$$f_0(t) = \int_{u \in \mathbf{R}} K(t, u) g(u)\, du, \tag{2}$$

and describe the general solution of (1).

3. Find a pair of integers (x, y) with $y \neq 0$ satisfying the equation

$$x^2 - 17y^2 = 1 .$$

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4. Let $f(z)$ be a holomorphic function on the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$, such that

$$|f(z)| < 1 \qquad \text{for all } z \in D$$

and

$$f(\tfrac{1}{2}) = f(\tfrac{-1}{2}) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{3}.$$

5. Let V be a finite-dimensional vector space over a field k, and assume that $\operatorname{char} k \neq 2$ and that $\dim V \geq 3$. Let $\bigwedge^2 V$ be the exterior square of V, that is, the k-vector space spanned by vectors of the form $\mathbf{v} \wedge \mathbf{w}$ for $\mathbf{v}, \mathbf{w} \in V$, subject to the relations ($\alpha \in k$):

$$\mathbf{v} \wedge \mathbf{w} = -(\mathbf{w} \wedge \mathbf{v}), \qquad (\alpha\mathbf{v}) \wedge \mathbf{w} = \alpha(\mathbf{v} \wedge \mathbf{w}), \qquad (\mathbf{v_1} + \mathbf{v_2}) \wedge \mathbf{w} = (\mathbf{v_1} \wedge \mathbf{w}) + (\mathbf{v_2} \wedge \mathbf{w}).$$

Let $T : V \to V$ be a linear transformation. T gives rise to a linear transformation

$$T_2 : \textstyle\bigwedge^2 V \to \bigwedge^2 V$$

defined by

$$T_2(\mathbf{v} \wedge \mathbf{w}) = (T\mathbf{v}) \wedge (T\mathbf{w}).$$

Show that if T_2 is multiplication by a non-zero scalar then so is T. Is this result true if $\dim V = 2$?

6. Let d be any integer and K any algebraically closed field. Recall that the set of plane curves $C \subset \mathbf{P}^2_K$ of degree d is parametrized by the projective space $\mathbf{P}^N_K$ of nonzero homogeneous polynomials $F \in K[X, Y, Z]$ of degree d modulo scalars ($N = \binom{d+2}{2} - 1$).

a) Let $\Delta \subset \mathbf{P}^N_K$ be the subset corresponding to singular curves. Show that for $d \geq 2$, Δ is a hypersurface in $\mathbf{P}^N_K$.

b) Let F and $G \in K[X, Y, Z]$ be linearly independent homogeneous polynomials of degree $d > 1$. For each point $t = [t_0, t_1] \in \mathbf{P}^1_K$, set $F_t = t_0 F + t_1 G$ and let $C_t \subset \mathbf{P}^2_K$ be the plane curve defined by the polynomial F_t. Show that for some $t \in \mathbf{P}^1_K$ the curve C_t is singular.

Let the field be of Characteristic Zero

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
Tuesday, March 12 (Day 1)

1. Let X be a compact n-dimensional differentiable manifold, and $Y \subset X$ a closed submanifold of dimension m. Show that the Euler characteristic $\chi(X \setminus Y)$ of the complement of Y in X is given by

$$\chi(X \setminus Y) = \chi(X) + (-1)^{n-m-1}\chi(Y).$$

Does the same result hold if we do not assume that X is compact, but only that the Euler characteristics of X and Y are finite?

2. Prove that the infinite sum

$$\sum_{p \text{ prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \ldots$$

diverges.

3. Let $h(x)$ be a C^∞ function on the real line $\mathbb{R}$. Find a C^∞ function $u(x,y)$ on an open subset of $\mathbb{R}^2$ containing the x-axis such that

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = u^2$$

and $u(x,0) = h(x)$.

4. a) Let K be a field, and let $L = K(\alpha)$ be a finite Galois extension of K. Assume that the Galois group of L over K is cyclic, generated by an automorphism sending α to $\alpha + 1$. Prove that K has characteristic $p > 0$ and that $\alpha^p - \alpha \in K$.

b) Conversely, prove that if K is of characteristic p, then every Galois extension L/K of degree p arises in this way. (Hint: show that there exists $\beta \in L$ with trace 1, and construct α out of the various conjugates of β.)

5. For small positive α, compute

$$\int_0^\infty \frac{x^\alpha \, dx}{x^2+x+1}.$$

For what values of $\alpha \in \mathbb{R}$ does the integral actually converge?

6. Let $M \in \mathcal{M}_n(\mathbb{C})$ be a complex $n \times n$ matrix such that M is similar to its complex conjugate $\overline{M}$; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\overline{M} = gMg^{-1}$. Prove that M is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday, March 13 (Day 2)

1. Prove the Brouwer fixed point theorem: that any continuous map from the closed n-disc $D^n \subset \mathbb{R}^n$ to itself has a fixed point.

2. Find a harmonic function f on the right half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ satisfying

$$\lim_{x \to 0+} f(x+iy) = \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0 \end{cases}.$$

3. Let n be any integer. Show that any odd prime p dividing n^2+1 is congruent to 1 (mod 4).

4. Let V be a vector space of dimension n over a finite field with q elements.

a) Find the number of one-dimensional subspaces of V.

b) For any $k : 1 \le k \le n-1$, find the number of k-dimensional subspaces of V.

5. Let K be a field of characteristic 0. Let $\mathbb{P}^N$ be the projective space of homogeneous polynomials $F(X,Y,Z)$ of degree d modulo scalars ($N = d(d+3)/2$). Let $W \subset \mathbb{P}^N$ be the subset of polynomials F of the form

$$F(X,Y,Z) = \prod_{i=1}^{d} L_i(X,Y,Z)$$

for some collection of linear forms $L_1, \ldots, L_d$.

a. Show that W is a closed subvariety of $\mathbb{P}^N$.

b. What is the dimension of W?

c. Find the degree of W in case $d = 2$ and in case $d = 3$.

6. a. Suppose that $M \to \mathbb{R}^{n+1}$ is an embedding of an n-dimensional Riemannian manifold (i.e., M is a hypersurface). Define the *second fundamental form* of M.

b. Show that if $M \subset \mathbb{R}^{n+1}$ is a compact hypersurface, its second fundamental form is positive definite (or negative definite, depending on your choice of normal vector) at at least one point of M.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Thursday, March 14 (Day 3)

1. In $\mathbb{R}^3$, let S, L and M be the circle and lines

$$\begin{aligned} S &= \{(x,y,z) : x^2 + y^2 = 1;\ z = 0\} \\ L &= \{(x,y,z) : x = y = 0\} \\ M &= \{(x,y,z) : x = \tfrac{1}{2};\ y = 0\} \end{aligned}$$

respectively.

a. Compute the homology groups of the complement $\mathbb{R}^3 \setminus (S \cup L)$.

b. Compute the homology groups of the complement $\mathbb{R}^3 \setminus (S \cup L \cup M)$.

2. Let $L, M, N \subset \mathbb{P}^3_{\mathbb{C}}$ be any three pairwise disjoint lines in complex projective threespace. Show that there is a unique quadric surface $Q \subset \mathbb{P}^3_{\mathbb{C}}$ containing all three.

3. Let G be a compact Lie group, and let $\rho : G \to GL(V)$ be a representation of G on a finite-dimensional $\mathbb{R}$-vector space V.

a) Define the *dual representation* $\rho^* : G \to GL(V^*)$ of V.

b) Show that the two representations V and V^* of G are isomorphic.

c) Consider the action of $SO(n)$ on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and the corresponding representation of $SO(n)$ on the vector space V of C^∞ $\mathbb{R}$-valued functions on S^{n-1}. Show that each nonzero irreducible $SO(n)$-subrepresentation $W \subset V$ of V has a nonzero vector fixed by $SO(n-1)$, where we view $SO(n-1)$ as the subgroup of $SO(n)$ fixing the vector $(0, \ldots, 0, 1)$.

4. Show that if K is a finite extension field of $\mathbb{Q}$, and A is the integral closure of $\mathbb{Z}$ in K, then A is a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$ (the degree of the field extension). (Hint: sandwich A between two free $\mathbb{Z}$-modules of the same rank.)

5. Let n be a nonnegative integer. Show that

$$\sum_{\substack{0 \le k \le l \\ k+l=n}} (-1)^l \binom{l}{k} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ -1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{if } n \equiv 2 \pmod 3 \end{cases}.$$

(Hint: Use a generating function.)

6. Suppose K is integrable on $\mathbb{R}^n$ and for $\epsilon > 0$ define

$$K_\epsilon(x) = \epsilon^{-n} K(\frac{x}{\epsilon}).$$

Suppose that $\int_{\mathbb{R}^n} K = 1$.

a. Show that $\int_{\mathbb{R}^n} K_\epsilon = 1$ and that $\int_{|x|>\delta} |K_\epsilon| \to 0$ as $\epsilon \to 0$.

b. Suppose $f \in L^p(\mathbb{R}^n)$ and for $\epsilon > 0$ let $f_\epsilon \in L^p(\mathbb{R}^n)$ be the convolution

$$f_\epsilon(x) = \int_{y \in \mathbb{R}^n} f(y) K_\epsilon(x-y)\, dy.$$

Show that for $1 \le p < \infty$ we have

$$\|f_\epsilon - f\|_p \to 0 \text{ as } \epsilon \to 0.$$

c. Conclude that for $1 \le p < \infty$ the space of smooth compactly supported functions on $\mathbb{R}^n$ is dense in $L^p(\mathbb{R}^n)$.

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Tuesday, October 24, 1995 (Day 1)

1. Let K be a field of characteristic 0.

a. Find three nonconstant polynomials $x(t), y(t), z(t) \in K[t]$ such that

$$x^2 + y^2 = z^2$$

b. Now let n be any integer, $n \geq 3$. Show that there do not exist three nonconstant polynomials $x(t), y(t), z(t) \in K[t]$ such that

$$x^n + y^n = z^n.$$

2. For any integers k and n with $1 \leq k \leq n$, let

$$S^n = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \ldots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$$

be the n-sphere, and let $D_k \subset \mathbb{R}^{n+1}$ be the closed disc

$$D_k = \{(x_1, \ldots, x_{n+1}) : x_1^2 + \ldots + x_k^2 \leq 1;\ x_{k+1} = \ldots = x_{n+1} = 0\} \subset \mathbb{R}^{n+1}.$$

Let $X_{k,n} = S^n \cup D_k$ be their union. Calculate the cohomology ring $H^*(X_{k,n}, \mathbb{Z})$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any C^∞ map such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv 0.$$

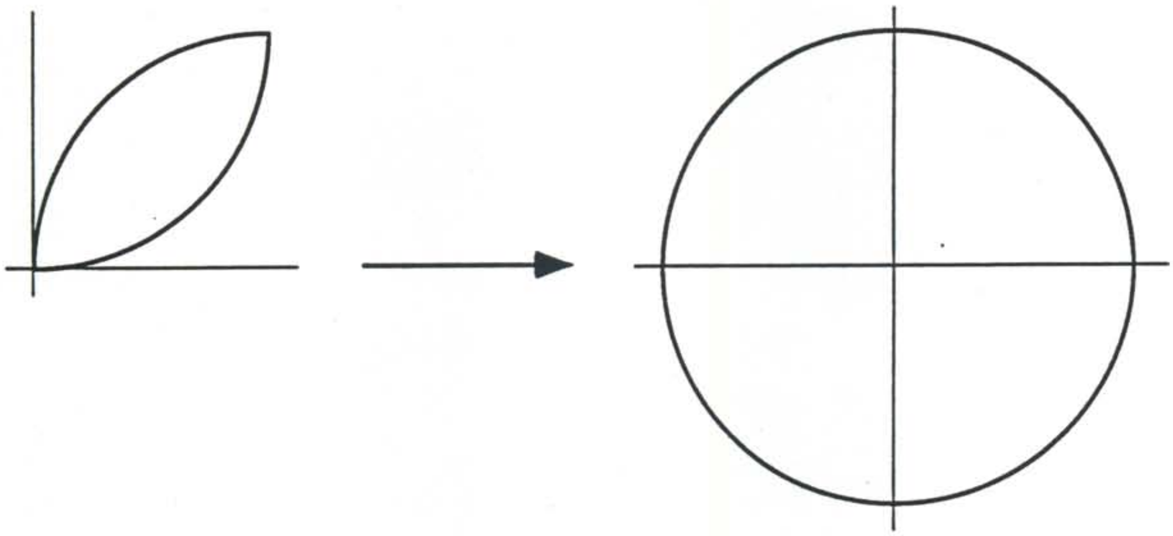
Show that if f is not surjective then it is constant.

4. Let G be a finite group, and let $\sigma, \tau \in G$ be two elements selected at random from G (with the uniform distribution). In terms of the order of G and the number of conjugacy classes of G, what is the probability that σ and τ commute? What is the probability if G is the symmetric group S_5 on 5 letters?

5. Let $\Omega \subset \mathbb{C}$ be the region given by

$$\Omega = \{z : |z-1| < 1 \quad \text{and} \quad |z-i| < 1\}.$$

Find a conformal map $f : \Omega \to \Delta$ of Ω onto the unit disc $\Delta = \{z : |z| < 1\}$.



6. Find the degree and the Galois group of the splitting fields over $\mathbb{Q}$ of the following polynomials:

a. $x^6 - 2$
b. $x^6 + 3$

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
Wednesday, October 25, 1995 (Day 2)

1. Find the ring A of integers in the real quadratic number field $K = \mathbb{Q}(\sqrt{5})$. What is the structure of the group of units in A? For which prime numbers $p \in \mathbb{Z}$ is the ideal $pA \subset A$ prime?

2. Let $U \subset \mathbb{R}^2$ be an open set.

a. Define a *Riemannian metric* on U.

b. In terms of your definition, define the *distance* between two points $p, q \in U$.

c. Let $\Delta = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$, and consider the metric on Δ given by

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

Show that Δ is complete with respect to this metric.

3. Let K be a field of characteristic 0. Let $\mathbb{P}^N$ be the projective space of homogeneous polynomials $F(X, Y, Z)$ of degree d modulo scalars ($N = d(d+3)/2$). Let U be the subset of $\mathbb{P}^N$ of polynomials F whose zero loci are smooth plane curves $C \subset \mathbb{P}^2$ of degree d, and let $V \subset \mathbb{P}^N$ be the complement of U in $\mathbb{P}^N$.

a. Show that V is a closed subvariety of $\mathbb{P}^N$.

b. Show that $V \subset \mathbb{P}^N$ is a hypersurface.

c. Find the degree of V in case $d = 2$.

d. Find the degree of V for general d.

4. Let $\mathbb{P}^n_{\mathbb{R}}$ be real projective n-space.

a. Calculate the cohomology ring $H^\bullet(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$.

b. Show that for $m > n$ there does not exist an *antipodal* map $f : S^m \to S^n$, that is, a continuous map carrying antipodal points to antipodal points.

5. Let V be any continuous nonnegative function on $\mathbb{R}$, and let $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$H(f) = \frac{-1}{2}\frac{d^2 f}{dx^2} + V \cdot f.$$

a. Show that the eigenvalues of H are all nonnegative.

b. Suppose now that $V(x) = \frac{1}{2}x^2$ and f is an eigenfunction for H. Show that the *Fourier transform*

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-ixy} f(x)dx$$

is also an eigenfunction for H.

6. Find the Laurent expansion of the function

$$f(z) = \frac{1}{z(z+1)}$$

valid in the annulus $1 < |z-1| < 2$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday, October 26, 1995 (Day 3)

1. Evaluate the integral

$$\int_0^\infty \frac{\sin x}{x} dx.$$

2. Let p be an odd prime, and let V be a vector space of dimension n over the field $\mathbb{F}_p$ with p elements.

   a. Give the definition of a *nondegenerate quadratic form* $Q : V \to \mathbb{F}_p$

   b. Show that for any such form Q there is an $\epsilon \in \mathbb{F}_p$ and a linear isomorphism

$$\begin{aligned} \phi : V &\longrightarrow \mathbb{F}_p^n \\ v &\longmapsto (x_1, \ldots, x_n) \end{aligned}$$

such that Q is given by the formula

$$Q(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \epsilon x_n^2$$

   c. In what sense is ϵ determined by Q?

3. Let G be a finite group. Define the *group ring* $R = \mathbb{C}[G]$ of G. What is the center of R? How does this relate to the number of irreducible representations of G? Explain.

4. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be any isometry, that is, a map such that the euclidean distance between any two points $x, y \in \mathbb{R}^n$ is equal to the distance between their images $\phi(x), \phi(y)$. Show that ϕ is *affine linear*, that is, there exists a vector $b \in \mathbb{R}^n$ and an orthogonal matrix $A \in O(n)$ such that for all $x \in \mathbb{R}^n$,

$$\phi(x) = Ax + b.$$

5. Let G be a finite group, $H \subset G$ a proper subgroup. Show that the union of the conjugates of H in G is not all of G, that is,

$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

Give a counterexample to this assertion with G a compact Lie group.

6. Show that the sphere S^{2n} is not the underlying topological space of any Lie group.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
March 15, 1995 (Day 1)

1. Describe the structure of the group of automorphisms of a cyclic group of order 600.

2.

a) Find the Fourier series for the periodic function given by

$$\begin{aligned} f(t) &= t && \text{for} \quad -\pi < t \leq \pi \\ f(2\pi + t) &= f(t) && \text{for all} \quad t\ . \end{aligned}$$

b) Deduce from this series the value of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \quad .$$

3. Let f be a holomorphic function on a domain containing the closed disc $\{z : |z| \leq 3\}$, such that

$$f(1) = f(i) = f(-1) = f(-i) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|.$$

Find all such functions f for which equality holds in this inequality.

(continued other side)

4. Describe the geodesics on the upper half plane relative to the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

5. Let f and g be entire holomorphic functions satisfying the identity

$$f(z)^2 = g(z)^6 - 1$$

for all $z \in \mathbb{C}$. Show that f and g are constant. Would the same conclusion be valid if f and g were assumed to be entire meromorphic functions?

Hint: think about the algebraic curve $y^2 = x^6 - 1$.

6. Let Y be a metric space.

a) Show that every open covering of Y has a finite subcovering if and only if every infinite sequence in Y has a convergent subsequence.

b) Is this true in general (i.e., non-metric) topological spaces? If not, give a counter-example.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
March 16, 1995 (Day 2)

1.) Prove the Banach fixed point theorem:

If (M, d) is a complete metric space, $f : M \to M$, and there exists a $0 < k < 1$ such that

$$d(f(x), f(y)) \leq k\, d(x, y).$$

Then f has a unique fixed point.

What can we say if we assume only that

$$d(f(x), f(y)) < d(x, y) \qquad \forall x \neq y?$$

2.

a) Let $f : X \to Y$ be a surjective algebraic map of two smooth projective curves over $\mathbf{C}$. What is the relation between the genus of X and Y and the branching of f.

b) Show that the curve $x^3 + y^3 + z^3 = 0$ in $\mathbf{C}P^2$ is homeomorphic to a torus by explicitly constructing a projection to $\mathbf{C}P^1$ and using part a).

3. Let Δ^4 be the standard 4-simplex:

$$\Delta^4 = \left\{ (x_0, x_1, x_2, x_3, x_4) \in \mathbf{R}^5 \middle| x_i \geq 0\ \forall i, \quad \sum_{i=0}^{4} x_i = 1 \right\}.$$

Let X be its 2-skeleton:

$$X = \left\{ (x_0, x_1, x_2, x_3, x_4) \in \Delta^4 \middle| x_i = x_j = 0 \quad \text{for some} \quad 0 \leq i < j \leq 4 \right\}.$$

Compute $\pi_1(x)$ and $\pi_2(X)$.

(continued other side)

Day 2, 2/22/95

N6

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is infinitely differentiable and satisfies

$$\lim_{n\to\infty} \frac{|f^{(n)}(x)|}{n!} = 0 \qquad \text{and} \qquad f^{(n)}(0) = 0$$

for all $x \in \mathbf{R}$ and all $n \geq 1$. Prove f is constant.

5. Let θ be a complex root of the equation

$$x^3 - 3x + 3 = 0.$$

a) Prove that the additive group of the ring $\mathbf{Z}[\theta]$ is finitely generated. Is it a free $\mathbf{Z}$-module?

b) Prove that the additive group of the ring $\mathbf{Z}[\theta^{-1}]$ is not finitely generated.

6. Find a conformal map of the slit unit disc

$$\{z : 0 < |z| < 1, \quad 0 < \arg z < 2\pi\}$$

to the unit disc $\{w : |w| < 1\}$.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
March 17,1995 (Day 3)

1. Find a homogeneous linear differential equation of order two of which $y = e^x$ and $y = x + x^2$ are two independent solutions.

2. Compute:

$$\int_0^\infty \frac{(\log x)^2 dx}{(1+x^2)}.$$

3.

a) Let $f(t) = a_m t^m + \cdots + a_0$ and $g(t) = b_n t^n + \cdots + b_0$ be polynomials of degree m and n with coefficients in a field F. Show that f and g have a common factor if and only if the determinant of the $(m+n)\times(m+n)$ matrix below vanishes:

$$\det \begin{pmatrix} a_0 & a_1 & \cdot & \cdot & \cdot & a_m & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & a_0 & \cdot & \cdot & \cdot & a_{m-1} & a_m & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & & \vdots & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_m \\ b_0 & b_1 & \cdot & \cdot & \cdot & b_{n-1} & b_n & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & b_0 & b_1 & \cdot & \cdot & b_{n-2} & b_{n-1} & b_n & 0 & \cdot & \cdot & 0 \\ \vdots & & & & & & \vdots & & & & & \vdots \\ 0 & 0 & \cdot & \cdot & b_0 & b_1 & b_2 & b_3 & \cdot & \cdot & \cdot & b_n \end{pmatrix} = 0.$$

b) Do this for a pair of polynomials with coefficients in an integral domain R.

c) Let $f(x,y)$ and $g(x,y)$ be polynomials in two variables over C, having no common factors. Show that f and g have at most a finite number of common zeroes.

(continued other side)

Day 3, 2/23/95

4. Construct a noncommutative group of order 8, and compute its character table.

5. Let S^2 be the 2-sphere, $T = S^1$ the torus, and Z the compact orientable surface of genus 2.

a) Does there exist a continuous map $f : S^2 \to T$ not homotopic to a constant?

b) Does there exist a continuous map $f : S^2 \to Z$ not homotopic to a constant?

c) Does there exist a continuous map $f : T \to Z$ not homotopic to a constant?

Justify your answers.

6. Let $X \subset \mathbf{P}^n$ be an ℓ-dimensional projective variety. Denote by G the Grassmannian $\mathsf{G}(k, n)$ of k-planes in $\mathbf{P}^n$, and let $Z \subset \mathsf{G}$ be the subset of k-planes meeting X, i.e.,

$$Z = \{\Lambda \in \mathsf{G} : \Lambda \cap X \neq \emptyset\}.$$

a). Show that Z is a closed subvariety of G.

b). What is the dimension of Z?

c). Show that Z is irreducible if and only if X is.

d). In case X is simply a linear subspace of $\mathbf{P}^n$, what is the singular locus of Z?

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
October 11, 1994 (Day 1)

1. Let $V \subset W$ be complex vector spaces of dimensions k and n respectively, and $\mathbf{P}V \subset \mathbf{P}W$ the corresponding projective spaces of one-dimensional subspaces of V and W. Find the cohomology ring $H^*(X, \mathbf{Z})$ of the complement $X = \mathbf{P}W \setminus \mathbf{P}V$.

2. Let K be an algebraically closed field of characteristic $p > 0$, and let $q = p^f$.

a). Show that the solutions of the equation $z^q = z$ form a subfield $F \subset K$.

b). How many solutions of the equation

$$x^2 - y^2 = 1$$

are there with $x, y \in F$? (Be careful to distinguish the case $p = 2$.)

3. Let $C^0([0,1])$ be the Banach space of continuous functions on $[0,1]$ with norm

$$\|f - g\| = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Let $K : [0,1] \times [0,1] \to \mathbf{R}$ obey

$$\sup_{(x,y)} (|K|(x,y) + |\frac{\partial}{\partial x} K|(x,y)) \leq 10.$$

Define $H : C^0([0,1]) \to C^0([0,1])$ by

$$(Hf)(x) = \int_0^1 K(x,y) f(y) dy.$$

(a) Prove that H is a bounded operator.

(b) Let $\{f_i\}_{i=1}^{\infty} \subset C^0([0,1])$ be a bounded sequence. Prove that $\{Hf_i\}_{i=1}^{\infty}$ has a convergent subsequence.

Continued other side

4. Give examples of

1) An integral domain R which is *not* integrally closed in its quotient field K, and its integral closure $\overline{R}$ in K.
2) An Artinian local ring of length n, for any $n \geq 1$ in $\mathbf{Z}$.
3) Two non-zero $\mathbf{Z}$-modules M and N, with $M \otimes N = 0$.
4) A discrete valuation ring with quotient field $\mathbf{Q}$.

5. Show, by the method of residues, that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \qquad a > b > 0.$$

Evaluate

$$\int_0^{\infty} \frac{\sin t}{t} dt.$$

6. a). Show that a compact, connected complex Lie group is abelian.

b). Describe such groups.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
October 12, 1994 (Day 2)

1. Let $\mathbf{Z}/n\mathbf{Z}$ be the cyclic group of order $n \geq 1$.
$(\mathbb{Z}/n\mathbb{Z})^*$

   a) Show that the automorphism group $A = \mathrm{Aut}(\mathbf{Z}/n\mathbf{Z})$ is abelian, and determine its structure.

   b) What is the order of the automorphism group of the finite group $G = \mathbf{Z}/5 \oplus \mathbf{Z}/5$? Is the group $\mathrm{Aut}(G)$ abelian?

   $GL(2, \mathbb{Z}/5)$ $\quad (5\times5-1)(5\times5-5) = 24\cdot 20 = 480$

   No.

2. Let f be a continuous complex valued function on $\mathbf{R}/\mathbf{Z}$ with Fourier expansion
$f(x) = \sum_{n\in\mathbf{Z}} a_n e^{2\pi i n x}$.

   a) Given an integral formula for the Fourier coefficients a_n.

   b) If f has p continuous derivatives, show that for $n \neq 0$, $|a_n| < \frac{C_f}{|n|^p}$, where C_f is a constant depending on f.

3. Let $S \subset \mathbf{R}^3$ be a smooth surface.

   a) Define the *Gaussian curvature* of S at a point $p \in S$.

   b) Show that if S is compact, then it contains a point p of positive Gaussian curvature.

4. Let S_n be the symmetric group on n letters, and let α_n be the number of elements of S_n having at least one fixed point. What is the limit of the ratio $\alpha_n/n!$ as $n \to \infty$?

Continued other side

Oct. 12, 1994 Day 2

5. Let G be a finitely presented group.

a) Show that there is a topological space X with fundamental group $\pi_1(X) \cong G$.

b) Give an example of X in case $G = \mathbb{Z} * \mathbb{Z}$ is the free group on two generators.

c) How many connected, 2-sheeted covering spaces of X are there?

3

6. Let V be an n-dimensional complex vector space, and let $W = \mathrm{Sym}^2 V^*$ be the vector space of symmetric bilinear forms on V. Let $W_k \subset W$ be the subset of forms of rank at most k.

a) Show that W_k is an irreducible algebraic variety.

b) What is the dimension of W_k? $\frac{k(k+1)}{2} + (n-k)k$

c) What is the singular locus of W_k?

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
October 13, 1994 (Day 3)

1. Let $f(x)$ be an irreducible polynomial of degree 3 over $\mathbf{Q}$, which has precisely one real root.

√a) Show that the splitting field K of $f(x)$ is a Galois extension, with $\mathrm{Gal}(K/\mathbf{Q})$ isomorphic to the symmetric group on 3 letters.

√b) How many extensions L of $\mathbf{Q}$ of degree 3 are contained in K? Are any of these fields normal over $\mathbf{Q}$? 3 No

√ c) If α_1, α_2, and α_3 are the 3 roots of $f(x)$ in K and $\beta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_3)$, show that β^2 lies in $\mathbf{Q}$, but $\mathbf{Q}(\beta)$ is a quadratic extension of $\mathbf{Q}$.

√2. Find a differentiable function $f(x)$ satisfying the differential equation

$$\frac{d^2 f}{d^2 x} + 4\frac{df}{dx} + 4f(x) = 0$$

such that $f(0) = 0$ and $f'(0) = 1$.

3. It is a classical fact that a smooth cubic surface in projective space $\mathbf{P}^3$ (over an algebraically closed field) contains 27 lines. Show that by contrast a general quartic surface $S \subset \mathbf{P}^3$ contains no lines.

Continued other side

Oct. 13, 1994

Day 3

4. Let $\mathbf{P}^m_{\mathbf{R}}$ be projective space of dimension m over the real numbers $\mathbf{R}$. Describe the cohomology ring $H^*(\mathbf{P}^m_{\mathbf{R}} \times \mathbf{P}^n_{\mathbf{R}}, \mathbf{Z})$.

5.) ? a) Define the radical I of a commutative ring R, and prove that it is an ideal of R.

b) Let R be a subring of a commutative field K, and let P be a proper prime ideal of R. Put $S = R - P$. Show that the ring R_S of fractions with denominators in S has a unique maximal ideal M. How is the field R_S/M related to the integral domain R/P?
field of fractions

6. Assume that $f(z)$ is analytic in the region $0 < |z| < r$. Show that there are constants $a_n \in \mathbf{C}$ such that

$$f(z) = \sum_{n \in \mathbf{Z}} a_n z^n$$

uniformly in each closed annulus $\epsilon \leq |z| \leq r - \epsilon$. In terms of the coefficients a_n,

a) When does f have an essential singularity at $z = 0$?

b) When does f have a pole of order $N \geq 1$ at $z = 0$?

c) When does f have a removable singularity at $z = 0$?

d) When is f the derivative of a function $g(z)$ in the region $0 < |z| < r$. $a_{-1} = 0$

Spring 1994 Qualifying Examination — Day 1

Department of Mathematics
Harvard University

(1) Let X_g be a Riemann surface of genus $g = 0, 1, \ldots$.
   (a) Describe the universal cover of X_g.
   (b) List the homotopy classes of maps from S^2 into X_g.
   (c) List the homotopy classes of maps from X_g into S^2.

(2) Let G be a group.
   (a) Define the group $A = \mathrm{Aut}(G)$ and show that there is a group homomorphism $G \to A$ whose kernel is the center of G.
   (b) If $|A| = 1$, show that $|G| \leq 2$.

(3) Suppose a_i and λ_i $(1 \leq i \leq n)$ are positive real numbers, and that $\sum a_i = 1$. Prove that

$$\sum a_i \lambda_i^2 \geq \left(\sum a_i \lambda_i\right)^2,$$

with equality iff $\lambda_1 = \ldots = \lambda_n$.

(4) Find complex numbers a_n and $b_n (n \in Z)$ such that

$$\cot z = \sum_{n=-\infty}^{\infty} \left(\frac{a_n}{z - n\pi} + b_n\right)$$

with uniform and absolute convergence on compact subsets of $C - Z\pi$. Use this series to compute $\sum_{n=1}^{\infty} \frac{1}{n^4}$. (Hint: one of the many possibilities is to use the residue theorem and as $n \to \infty$ consider the integral of $\frac{f(w)dw}{w(w-z)}$ over the square C_n with side equal to $(2n+1)\pi$ centered at the origin, where $f(z)$ is an appropriate function constructed from $\cot z$.)

(5) Prove (a simple version of) the *Whitney embedding theorem*, saying that every compact smooth n-dimensional manifold M can be embedded in $\mathbf{R}^{2n+1}$, by first showing that M can be embedded in $\mathbf{R}^N$ for some large number N and then by iteratively reducing the necessary value of N.

(6) (a) What is the dimension of the projective space $\mathbf{P}^N$ of non-zero homogeneous polynomials of degree d in 3 variables, modulo scalars?
   (b) Show that the subset of polynomials $F(X, Y, Z)$ such that the zero locus $V(F) \subset \mathbf{P}^2$ of F is not a smooth plane curve of degree d is an algebraic subvariety of $\mathbf{P}^N$.

GOOD LUCK!

Spring 1994 Qualifying Examination — Day 2

Department of Mathematics
Harvard University

(1) (a) Let E be an arbitrary vector bundle on the n-dimensional simplex Δ^n. Show (from basic principles) that E is trivial.

(b) Let E be an $(n+1)$-dimensional vector bundle over S^n. Show that E admits a nowhere vanishing cross-section.

(2) Let k be a field and let $f(x)$ be a polynomial in $k[x]$. Let I be the ideal generated by $f(x)$ and let $A = k[x]/I$.

(a) When is A a field?

(b) When is A isomorphic to a product of fields?

(c) If $k = \mathbf{Z}/p\mathbf{Z}$ (with p a prime) and $f(x) = x^{p^3} - x$, show that A is isomorphic to a product of fields. Which fields occur and with what multiplicity?

(3) (a) Define the *Fourier transform* $\tilde{f}$ of an integrable function $f : \mathbf{R} \to \mathbf{C}$.

(b) Show that the Fourier transform of the function $f(x) = e^{-x^2/2}$ is proportional to that function itself.

(4) (a) Estimate the asymptotic behavior of a typical solution of the differential equation

$$t^2y'' + ty' + (3t^2 - 4)y = 0 \qquad (☺)$$

near $t = 0$. (Notice that $t = 0$ is a regular singular point of ☺).

(b) Notice, however, that on a codimension 1 subspace of the space of solutions of ☺, the asymptotic behavior is different. What is the asymptotic behavior of a non-zero solution belonging to this exceptional subspace?

(5) Let $p \in \mathbf{R}^n$ be a non-degenerate critical point of a smooth function $f : \mathbf{R}^n \to \mathbf{R}$, and let S be some very small sphere around p. Compute the degree of the normalized gradient map $g : S \to S^{n-1}$ defined by

$$x \mapsto \frac{\nabla f(x)}{|\nabla f(x)|}$$

in terms of some higher derivatives of f at p.

(6) Let $\mathbf{P}^n$ denote the n-dimensional projective space over some arbitrary ground field.

(a) Define the Segre map $\sigma : \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^{nm+n+m}$.

(b) Show that if $n = m = 1$, the image of σ is a smooth quadric (surface of degree 2) in $\mathbf{P}^3$.

GOOD LUCK!

Spring 1994 Qualifying Examination — Day 3

Department of Mathematics
Harvard University

(1) Let $\mathcal{C}$ be a finite complex of finite dimensional vector spaces,

$$\mathcal{C}: \quad 0 \to C_n \to C_{n-1} \to \ldots \to C_1 \to C_0 \to 0,$$

and let H_i denote the ith homology of $\mathcal{C}$.

(a) Define 'an endomorphism of $\mathcal{C}$'.

(b) Let $g : \mathcal{C} \to \mathcal{C}$ be an endomorphism of $\mathcal{C}$. Prove that

$$\sum_{i=0}^{n}(-1)^i \mathrm{tr}\; g|_{C_i} = \sum_{i=0}^{n}(-1)^i \mathrm{tr}\; g|_{H_i}.$$

(2) Find all natural numbers $n \geq 1$ such that $e^{2\pi i/n} \in \mathbf{C}$ lies in the field $\mathbf{Q}(i) = \{a + bi : a, b \in \mathbf{Q}\}$.

(3) Compute the integral

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}dx.$$

(4) Let $\langle e_1, e_2 \rangle$ be a basis for $\mathbf{C}$ over $\mathbf{R}$.

(a) If $f : \mathbf{C} \to \mathbf{C}$ is an entire function which satisfies $f(z + e_1) = f(z + e_2) = f(z)$ for all $z \in \mathbf{C}$, show that f is constant.

(b) Construct an example of a meromorphic non-constant function $f : \mathbf{C} \to \mathbf{CP}^1$ which satisfies $f(z + e_1) = f(z + e_2) = f(z)$ for all $z \in \mathbf{C}$.

(5) Let ω be closed non-degenerate 2-form on some smooth $2n$-dimensional manifold M.

(a) Let a function $f : M \to \mathbf{R}$ be smooth. Show that there exists a unique vector field X_f on M such that $i(X_f)\omega = -df$, where $i(X_f)$ denotes interior multiplication by X_f.

(b) Assume that for some function $f : M \to \mathbf{R}$ the corresponding vector field X_f integrates to a flow $\exp tX_f$. Show that the resulting flow preserves the volume form $\omega^n/n!$ corresponding to ω; i.e., show that

$$(\exp tX_f)^* \left(\frac{\omega^n}{n!}\right) = \frac{\omega^n}{n!}.$$

(c) A flow $\exp tX_f$ generated as above is called "a Hamiltonian flow", and the function f generating it is called "the Hamiltonian function". Let S^2 be the unit sphere in $\mathbf{R}^3$, and let ω be the standard volume form of S^2. Is rotation around the z-axis a Hamiltonian flow? If so, what is the corresponding Hamiltonian function?

(6) What is the dimension of the space of holomorphic forms on the Fermat curve

$$C = \{X^N + Y^N + Z^N = 0\} \subset \mathbf{P}^2 \;?$$

Bonus: How many rational points are there on C for $N \geq 3$?

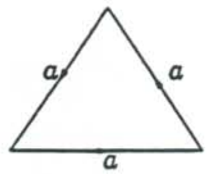
GOOD LUCK!

F 93-94

Fall 1993 Qualifying Examination — Day 1

Department of Mathematics
Harvard University

(1) Let Y be the space obtained from a triangle by identifying its sides in the following manner:



(all sides identified to each other)

(a) Compute (and justify your computation) the fundamental group of Y.
(b) Describe the universal cover $\tilde{Y}$ of Y in as simple terms as possible.
(c) Compute (and justify your computation) $\pi_2(Y)$.

(2) Let I and J be defined by the definite integrals

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx, \quad J = \int_0^1 x^4(1-x)^4 dx.$$

(a) Show that

$$0 < I < J = \frac{1}{630}.$$

(b) Evaluate I in closed form.
(c) What classical inequality follows?

(3) Define "orientable manifold" and prove that all Lie groups are orientable.

(4) Find with proof all rings A (commutative or not) for which $G = A - \{0\}$ is a finite group with A's multiplication as the group law, and determine the structure of all groups G which occur in this way.

(5) Let k be a field. Define the general linear group $GL(n,k)$. What is the center of $GL(n,k)$? Let $PGL(n,k)$ be the quotient of $GL(n,k)$ by its center. Define projective space $P^{n-1}(k)$ of dimension $(n-1)$ over k, and show that the group $PGL(n,k)$ acts 2-transitively on this space. Show that the action is triply transitive when $n = 2$.

(6) Prove the Schwarz lemma, saying that if D is the open unit disk in the complex plane and f is a holomorphic function on D such that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$, then $|f(z)| \leq |z|$ for all $z \in D$, $|f'(0)| \leq 1$, and if one of these inequalities is an equality then $f(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi)$.

GOOD LUCK!

?...-

Fall 1993 Qualifying Examination — Day 2

Department of Mathematics
Harvard University

(1) Let S^2 be the two dimensional sphere, and let M be the space of smooth closed curves in S^2 (a curve has to have non-zero velocity everywhere; self-intersections are allowed). To put a topology on M, parametrize all curves by normalized arc-length, then use C^∞ topology. How many connected components does M have?

(2) In an experiment performed last week in the Harvard University Mathematical Laboratories, one end of a metallic string of length 2π was held in ice and its other end was held in boiling water for a long time, until a stable equilibrium was achieved. Then the string was removed from the ice and the water, was bent to form a circle (and its two ends were thus in contact), and was put on a table made of a heat-insulating material.

 (a) If x measures the distance to the (initially) iced end of the string and t measures time, write the boundary conditions for the heat equation $u_t = u_{xx}$ which describe the above situation.

 (b) Solve the equations you just got.

 (c) At time $t = 20$, the string will have an almost constant temperature. What will this temperature be? Estimate the difference between the actual temperature of the string at time $t = 20$ and the constant you just found.

(3) A smooth vector field V on the three dimensional Euclidean space $\mathbf{R}^3$ is given by

$$V = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z},$$

where a, b, and c are smooth functions on $\mathbf{R}^3$. Determine (in terms of a, b, c, and their derivatives) when is the distribution P of planes orthogonal to V integrable.

cont...

Day 2 p 2
Fall 93

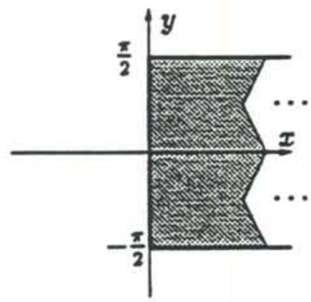
(4) Recall that two matrices A and B with entries in some field $\mathbf{F}$ are called *similar* over $\mathbf{F}$ if there exists invertible matrices M and N with entries in $\mathbf{F}$ for which $B = MAN$. Prove that two rational matrices which are similar over the complex numbers are similar over the rationals.

(5) Let $F(X,Y,Z)$ be a homogeneous cubic polynomial, and let C be the cubic curve $(F = 0)$ in the projective plane; assume that C is smooth. Let $H(X,Y,Z)$ be the Hessian of F, that is, the determinant of the 3×3 matrix of second partial derivatives of F. Show that a point P on C is a zero of H if and only if there is a line in the plane meeting C only at P.

(6) Find a conformal bijection between the semi-infinite strip

$$D = \{z \in \mathbf{C} : \mathrm{Re}(z) > 0, |\mathrm{Im}(z)| < \pi/2\}$$

outlined below and the unit disk in the complex plane.



GOOD LUCK!

Fall 1993 Qualifying Examination — Day 3

Department of Mathematics
Harvard University

(1) (a) Compute the homology of the complement of an embedded circle in $\mathbf{R}^3$.
(b) Do the same for a pair of disjoint embedded circles. Exhibit generators.
(c) Compute the homology of the complement of an embedded circle in $\mathbf{R}^4$, and show that the fundamental group of this complement is trivial.

(2) If $\{f_n : [0,1] \to \mathbf{R}\}$ is a sequence of continuous functions,

$$f_1(x) \geq f_2(x) \geq f_3(x) \geq \dots \geq 0$$

for all $x \in [0,1]$, and $f_n(x)$ tends to 0 pointwise, is the convergence necessarily uniform? (Prove or give a counterexample).

(3) (a) Define the degree of a smooth map $\phi : M \to N$, where both M and N are both compact orientable k-dimensional manifolds.
(b) Compute the degree of the self-map $g \mapsto g^q$ defined on the group $SU(2)$. q here is an arbitrary integer.

(4) Let k be the field $\mathbf{Q}[x]/x^3 - x^2 - 4x - 1$, whose discriminant d_k is equal to 169.
(a) Show that the extension $k/\mathbf{Q}$ is normal, is totally real, and has Galois group $\mathbf{Z}/3\mathbf{Z}$.
(b) Show that any two different roots of $x^3 - x^2 - 4x - 1$ generate a subgroup of finite index of the group of units of k.

(5) Define a hyperelliptic curve of genus $g > 1$. Prove that every curve of genus 2 is hyperelliptic, but not all curves of genus 3 are.

(6) The nth Catalan number C_n is the number of possible products of n generators in a free non-associative algebra. For example, $C_1 = 1$, $C_2 = 1$, $C_3 = 2$, and $C_4 = 5$, with the last equality because the possible products of a, b, c, d are $\{a(b(cd)), a((bc)d), (a(bc))d, ((ab)c)d, (ab)(cd)\}$.
(a) Prove that for $n > 1$,

$$C_n = C_1C_{n-1} + C_2C_{n-2} + \ldots + C_{n-1}C_1.$$

(b) Deduce that

$$\sum_{n=1}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2}.$$

(c) What is the number c for which the growth rate of c^n as $n \to \infty$ is the closest to that of C_n?

GOOD LUCK!

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
March 2, 1993 (Day 1)

NOTE: For any multi-part problem, you may receive partial credit for doing any part, assuming the statement of a preceding part.

1. How many groups of order 21 are there up to isomorphism?
   Describe them all.

2. a. State the Poincaré Duality and Künneth theorems.
   b. Find an example of a compact 4-manifold M such that
   $$\dim_{\mathbf{Q}}(H_1(M,\mathbf{Q})) \neq \dim_{\mathbf{Q}}(H_3(M,\mathbf{Q})).$$

3. a. Prove **Rouché's theorem**. Let $f(z)$ and $g(z)$ be functions analytic inside and on a simple closed curve $C \subset \mathbf{C}$. Suppose that $|g(z)| < |f(z)|$ everywhere on C. Show that $f(z)$ and $f(z)+g(z)$ have the same number of zeroes inside C.
   b. Prove that all roots of the polynomial $f(z) = z^7 - 5z^3 + 12$ lie in the annulus $1 < |z| < 2$.

4. Let $f(x,y)$ be a polynomial of degree 3 with coefficients in $\mathbf{Q}$, and suppose there exists a solution $f(a,b)=0$ with $a,b \in \mathbf{Q}$. Show that there exist infinitely many solutions $f(x,y)=0$ such that x and y are contained in a quadratic extension of $\mathbf{Q}$ (the quadratic extension may depend on (x,y)).

5. Let E be a Banach space over $\mathbf{C}$, E^* the space of bounded linear maps $E \to \mathbf{C}$. Recall that a sequence $\{u_j\}$ in E is said to *converge weakly* to u if for each $A \in E^*$ we have

$$\lim_{j \to \infty} (A(u_j)) = A(u).$$

Show that weakly convergent sequences are bounded. (Hint: use the fact that E^* is also a Banach space.)

6. Let $\alpha : I \to \mathbf{R}^3$ be any regular arc, $\mathbf{t}(u), \mathbf{n}(u)$, and $\mathbf{b}(u)$ its unit tangent, normal and binormal vectors at $\alpha(u)$. Consider the *normal tube of radius* ϵ around α, that is, the parametrized surface given by

$$\varphi(u,v) = \alpha(u) + \epsilon\cos(v)\cdot\mathbf{n}(u) + \epsilon\sin(v)\cdot\mathbf{b}(u).$$

a. For what values of ϵ is this an immersion?

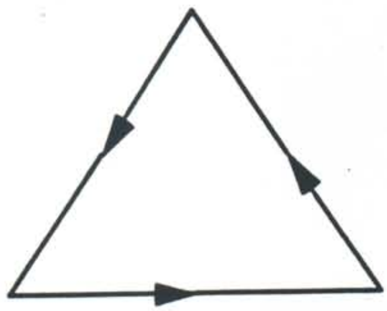
b, Assuming α itself has finite length, find the surface area of the normal tube of radius ϵ around α.

The answers to both questions should be expressed in terms of the curvature $\kappa(u)$ and torsion $\tau(u)$ of α.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
March 3, 1993 (Day 2)

1. Let X be the topological space obtained by identifying all three sides of a triangle as shown in the diagram



Compute the homology groups of X with coefficients in $\mathbf{Z}$, and with coefficients in $\mathbf{Z}/3\mathbf{Z}$. Is X a manifold?

2. Let K be the splitting field of the polynomial $x^4 - 2$ over $\mathbf{Q}$, and let G be the Galois group of K over $\mathbf{Q}$.
   a. Show that G is a dihedral group.
   b. Describe all subfields of K containing $\mathbf{Q}$.

3. Let X be a Riemann surface, $\mathbf{CP}^1$ the Riemann sphere, and suppose that $f : \mathbf{CP}^1 \to X$ is a nonconstant holomorphic map.
   Show that $X \cong \mathbf{CP}^1$.

4. Let $M \subset \mathbf{R}^3$ be a surface (i.e., submanifold of dimension 2) in $\mathbf{R}^3$, with the induced Riemannian metric, and let $\gamma : (0,1) \to M$ be a differentiable arc parametrized by arc length. Show that γ is a geodesic if and only if the acceleration vector $\gamma''(t)$ is perpendicular to the tangent space to M at $\gamma(t)$ for all t.

5. For a complex number s with $\mathrm{Re}(s) > 1$, define

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$

Prove that $|\zeta(2+it)| > 6/\pi^2$ for all real t.

6. Let O_n be the group of $n \times n$ orthogonal matrices. A **Haar** measure on O_n is a positive measure λ that is left invariant ($\lambda(\Gamma A) = \lambda(A)$ for each $\Gamma \in O_n$ and Borel measurable $A \subset O_n$) and normalized so that $\lambda(O_n) = 1$. Prove that

$$\int_{O_n} \Gamma_{1,1} \lambda(d\Gamma) = 0$$

and

$$\int_{O_n} (\Gamma_{1,1})^2 \lambda(d\Gamma) = 1/n$$

where $\Gamma_{i,j}$ is the $(i,j)^{\text{th}}$ entry of Γ.

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
March 4, 1993 (Day 3)

1. Find the Fourier transform $\hat{f}(t)$ of the function

$$f(x) = \frac{1}{1+x^2}\,.$$

2. If $y = f(x)$ is a differentiable function, **Newton's method** for finding roots $f(a) = 0$ consists in taking an initial guess x_0 and finding the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - f(x_n)/f'(x_n).$$

The hope is that this sequence has a limit which is a root.

If $f(x) = x^2 - 1$ and $x_0 > 0$ show that the method works and analyze the rate of convergence of $\{x_n\}$.

3. Let V be a 2-dimensional vector space over $\mathbf{C}$, $G = SL(V) \cong SL_2(\mathbf{C})$ be the group of automorphisms of V of determinant 1. Consider the action of G on the space W_n of homogeneous polynomials of degree n on V.

   a. Show that W_n is an irreducible representation of $SL(V)$.

   b. Show that every polynomial of degree n in two variables can be expressed as a sum of n^{th} powers of linear functions.

4. An $n \times n$ real matrix $M = (m_{i,j})$ is called *doubly stochastic* if it satisfies the conditions

$$m_{i,j} \geq 0 \quad \text{and} \quad \sum_{j=1}^{n} m_{i,j} = \sum_{i=1}^{n} m_{i,j} = 1 \quad \text{for all } i, j.$$

Let DS be the set of all doubly stochastic $n \times n$ matrices.

   a. Show that DS is a convex subset of $\mathbf{R}^{n^2}$.

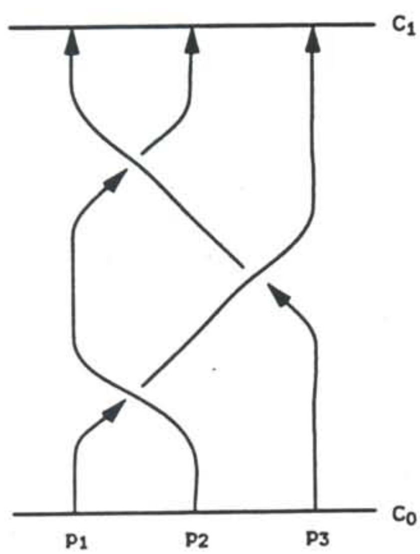
   b. Show that the extreme points of DS are the permutation matrices.

c. The *permanent* $per(M)$ of a matrix M is defined just like the determinant, only with all minus signs changed to pluses. Show that the permanent of a doubly stochastic matrix is always strictly positive.

5. a. Let $C \subset \mathbf{P}^n$ be an irreducible algebraic curve of degree n not contained in a hyperplane. Show that $C \cong \mathbf{P}^1$.

   b. Now suppose that $C \subset \mathbf{P}^n$ is an irreducible algebraic curve of degree $n+1$, again not contained in a hyperplane. What can you say about the genus of C?

6. A *k-braid* B is a subset of $\mathbf{C} \times I$ that looks something like:

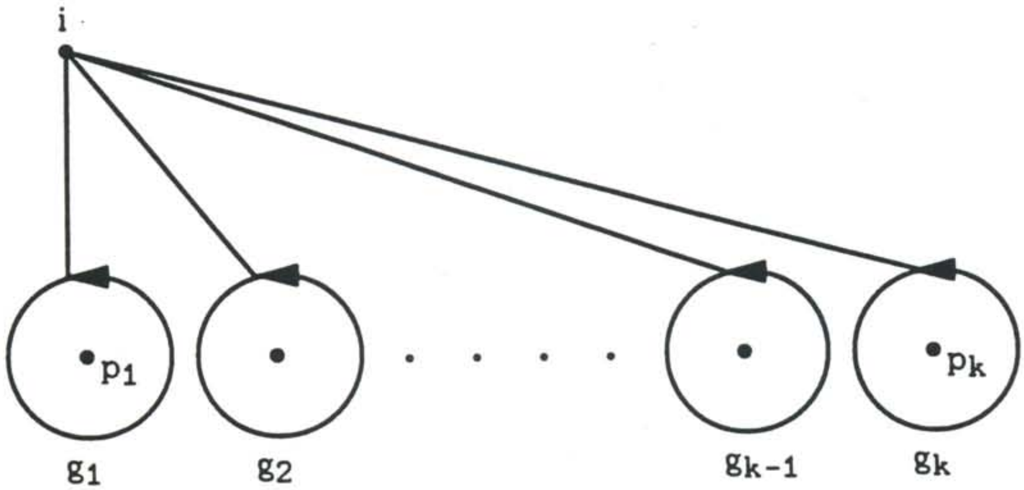


So at any "time slice" parametrized by a "time" $t \in I$ the braid looks like a configuration of k different points in $\mathbf{C}$. As time progresses, this configuration smoothly changes from some initial configuration $(p_1, \ldots, p_k)$ to some final configuration $(p_{\pi 1}, \ldots, p_{\pi k})$, obtained from the initial configuration via some permutation π of $1, \ldots, k$. In our example, π is the transposition that exchanges 1 and 3.

Let C_0 be the complement of the endpoints of the braid in $\mathbf{C} \times \{0\}$, let C_1 be the complement of the endpoints of the braid in $\mathbf{C} \times \{1\}$, and let B^c be the complement of the braid B in $\mathbf{C} \times I$.

(1) Show that the inclusions $C_0 \hookrightarrow B^c$ and $C_1 \hookrightarrow B^c$ induce isomorphisms of fundamental groups.

(2) Notice that $\pi_1(C_0) = \pi_1(C_1) = F_k$ is the free group on k generators $g_1, \ldots, g_k$, where the generators can be taken to be as the figure below, assuming the points $p_1, \ldots, p_k$ are just the integers $1, \ldots, k$, and the basepoint is i:



Let ξ_B be the composition $\pi_1(C_0) \to \pi_1(B^c) \to \pi_1(C_1)$, regarded as an automorphism of F_k. Prove that ξ_B maps g_i to a conjugate of $g_{\pi i}$ for every $1 \le i \le k$, and maps the product $g_1 g_2 \cdots g_k$ to itself.

(3) Compute $\xi_B(g_2)$ in the case of the braid displayed above.

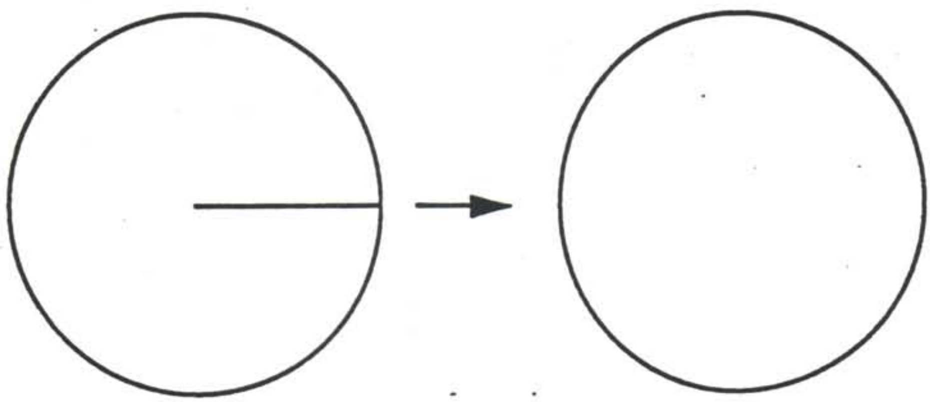
QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
October 6, 1992 (Day 1)

92-93

1. Find a conformal map (that is, a bijective holomorphic map) from the slit unit disc

$$\Omega = \{z : 0 < |z| < 1,\ 0 < arg(z) < 2\pi\}$$

to the unit disc $\Delta = \{z : |z| < 1\}$.



2. Let $X \subset \mathbb{R}^3$ be a differentiable surface.

   a. Define the *Gaussian curvature* of X at a point p.

   b. Calculate the Gaussian curvature of the hyperboloid

$$X = \{(x, y, z) : z^2 = x^2 + y^2 - 1\}.$$

3. Let X be a compact orientable surface of genus g.
   a. How many connected covering spaces of degree 2 does X have?
   b. Show that these coverings are all homeomorphic to one another as topological spaces.

4. Find the solution of the differential equation

$$\frac{dy}{dx} - 2xy = x$$

satisfying $y(0) = 1$.

5. Let G be a finite group of affine transformations of $\mathbf{R}^n$, that is, maps of the form $v \longmapsto Av + b$ with $A \in GL_n(\mathbf{R})$ and $b \in \mathbf{R}^n$. Show that there is a point of $\mathbf{R}^n$ fixed by every element of G.

6. Let A, B, C and D be any 3×3 symmetric matrices of complex numbers. Show that there exist complex numbers α, β, γ and δ, not all zero, such that the linear combination

$$M = \alpha A + \beta B + \gamma C + \delta D$$

has rank 1 or less.

(

QUALIFYING EXAMINATION
Harvard University
Department of Mathematics
October 7, 1992 (Day 2)

1. a. Let $\Gamma \subset \mathbf{R}^3$ be the union of the circle $x^2 + y^2 - 1 = z = 0$ and the line $x = y = 0$. Find the homology and homotopy groups of the complement $\mathbf{R}^3 - \Gamma$.

   b. Similarly, let $\Omega \subset \mathbf{R}^5$ be the union of the sphere $x^2 + y^2 + z^2 - 1 = u = v = 0$ and the plane $x = y = z = 0$. Find the homology groups of the complement $\mathbf{R}^5 - \Omega$.

2. Let G and H be connected topological groups and $\varphi : G \to H$ a continuous homomorphism. If φ is a covering space map, show that the kernel of φ is contained in the center of G.

3. Let k be any field, $L = k(t)$ a purely transcendental extension and K the subfield $K = k(t^n) \subset L$. Analyze the extension $K \subset L$ from the point of view of Galois theory. In particular, under what conditions is this a Galois extension?

4. Let $f(x, y) \in \mathbf{R}[x, y]$ be a polynomial with real coefficients, and suppose that

$$f(t.e^{-t^2}) = 0 \qquad \forall t \in \mathbf{R}.$$

Prove that $f(x, y) \equiv 0$.

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5. a. Find the Taylor series expansion of the function

$$g(z) = \frac{z}{z^3 - z^2 - z + 1}$$

around the origin $z = 0$.

b. Find the Laurent series expansion of the function

$$f(z) = \frac{z}{(z-1)(z-2)}$$

valid in the annulus $\{z : 1 < |z| < 2\}$.

6. Let H be a Hilbert space and $K : H \to H$ a compact operator. Set $T = I + K$, where I denotes the identity.

(i) Show that $\dim \ker T < \infty$;

(ii) Show that there exists a constant C such that for all $f \in (\ker T)^{\perp}$, $\|f\| \leq C\|Tf\|$;

(iii) Use (ii) to show that the range of T is closed;

(iv) Show that *either* the equation $Tf = g$ has a unique solution f for every $g \in H$, *or* it has no solutions for some g and infinitely many for others.

92-93

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
October 8, 1992 (Day 3)

1. Let p be a real number, $0 < p < 1$. Evaluate the integral

$$\int_0^\infty \frac{x^{p-1}dx}{1+x}.$$

2. Let X be a compact orientable surface of genus 2, and let $\varphi : X \to X$ be a fixed-point-free homeomorphism of finite order.

   a. Show that φ is of order 2 and orientation-reversing.

   b. Show that such homeomorphisms of surfaces of genus 2 do exist.

3. Let $X \subset \mathbf{P}^n$ be a k-dimensional projective variety. Denote by $\mathbf{P}^{n*}$ the dual projective space of hyperplanes $H \cong \mathbf{P}^{n-1} \subset \mathbf{P}^n$, and consider the *universal hyperplane section*

$$\Gamma_X = \{(p, H) : p \in H\} \subset X \times \mathbf{P}^{N*}.$$

   a. Show that Γ_X is a closed subvariety of the product $X \times \mathbf{P}^{n*}$.

   b. Find the dimension of Γ_X.

   c. Show that Γ_X is irreducible if and only if X is.

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4. Let S_4 be the symmetric group on 4 letters.

   a. Show that S_4 is the group of symmetries of an oriented cube – that is, the group of orientation preserving euclidean motions of $\mathbf{R}^3$ carrying a cube $\Omega \subset \mathbf{R}^3$ into itself.

   b. Describe the irreducible representations of S_4 and their characters.

   c. How does the permutation representation of S_4 on the vertices of the cube Ω decompose into a direct sum of irreducible representations?

5. a. Show that the sum $\sum 1/p$, where p ranges over all primes, does not converge.

   b. Show that the sum $\sum \frac{(-1)^{\frac{p-1}{2}}}{p}$, where p ranges over all odd primes, does converge.

6. Let S_n be the symmetric group on n letters. Let $f : S_n \to \mathbf{Z}$ be the map whose value on any permutation $\sigma \in S_n$ is 1 if σ has a fixed point and 0 otherwise and $g : S_n \to \mathbf{Z}$ the map whose value on any σ is the number of fixed points of σ. Let a_n and b_n be the averages of the functions f and g, that is, the probability that a random permutation has a fixed point and the expected number of fixed points it has. Find the limit as n approaches ∞ of a_n and b_n.

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Spring 92

(91-92)

No Quals Given

F 91-92

QUALIFYING EXAMINATION

Harvard University
Department of Mathematics
October 8, 1991 (Day 1)

1. Think of $S^1 \times S^1$ as $\mathbf{R}^2/\mathbf{Z}^2$.

a) What is $\pi_1(S^1 \times S^1)$?

b) Define X to be the space obtained by taking

$[0,1] \times (S^1 \times S^1)$ and identifying

$(0,(y_1,y_2))$ with $(1,(-y_2,y_1))$.

a) Compute $H_1(X)$.

b) Compute $\pi_1(X)$.

c) Is X homeomorphic to $\mathbf{RP}^2 \times S^1$? Justify your answer.

2. Fermat proved that if p is prime, then $a^p \equiv a \pmod{p}$ for all a.

Generalize this to prove the following:

Let $f_n(a) = \sum_{d|n} \mu(d) a^{n/d}$ with $\mu(d)$ defined as zero if d is not square free and $\mu(d) = 1$ or -1 as d has an even or odd number of prime divisors otherwise. Thus, $f_6(a) = a^6 - a^3 - a^2 + a$.

Prove that for all positive integers n and all integers a,

$$f_n(a) \equiv 0 \pmod{n}.$$

3. Let $e : (0,1) \longrightarrow \mathbf{R}^2$ be an arc parametrized by arc length, i.e., such that the derivative $e'(t)$ has norm 1 for all t. We call $v(t) = e'(t)$ the ***unit tangent vector*** to the arc e at time t, and define both the unit vector $n(t)$, called the ***unit normal vector***, and the scalar $\kappa(t) \geq 0$, called the *curvature*, by the equation

$$v'(t) = \kappa(t) \cdot n(t).$$

a) Assume $\kappa(t) > 0$ and show that $n(t)$ is orthogonal to $v(t)$ for all t, and that the derivative of n is given by

$$n'(t) = -\kappa(t) \cdot v(t).$$

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b. Again assuming $\kappa(t) > 0$, we define the *osculating circle* to the arc e at time t to be the circle of radius $1/\kappa(t)$ with center $e(t) + n(t)/\kappa(t)$ – that is, a circle of radius $1/\kappa(t)$ tangent to the arc at $e(t)$. Show that *if the curvature function $\kappa(t)$ is monotone, then the osculating circles to the arc are nested.*

4. Let $f, g : \mathbf{C} \to \mathbf{C}$ be analytic on a simply connected region containing a simple closed curve γ. Suppose that γ misses all the zeros of f and g, and that on γ

$$|f(z) - g(z)| < |f(z)|.$$

Show that f and g have the same number of zeros, counting multiplicity, inside γ, i.e., prove Rouche's theorem.

5. Let (X, d) be a compact metric space and suppose f is a map $X \to X$ such that for every $x \neq y$,

$$d(f(x), f(y)) < d(x, y).$$

Show that there exists a unique $x_0 \in X$ satisfying $f(x_0) = x_0$.

6. a) Let $V = \mathrm{Hom}(\mathbf{C}^m, \mathbf{C}^n)$ be the vector space of $m \times n$ matrices. and let $\Lambda \subset V$ be any linear subspace such that for all $A \in \Lambda$, the rank of A is at most 1. Show that either

   i) there exists a hyperplane $U \cong \mathbf{C}^{m-1} \subset \mathbf{C}^m$ such that $U \subset \ker(A)$ for all $A \in \Lambda$;

   or

   ii) there exists a line $W \cong \mathbf{C} \subset \mathbf{C}^n$ such that $\mathrm{Im}(A) \subset W$ for all $A \in \Lambda$.

   b) Let $s : \mathbf{P}_{\mathbf{C}}^{m-1} \times \mathbf{P}_{\mathbf{C}}^{n-1} \longrightarrow \mathbf{P}_{\mathbf{C}}^{mn-1}$ be the Segre map given by

$$s : ([Z_1, \cdots, Z_m], [W_1, \cdots, W_n]) \longmapsto [\cdots, Z_i W_j, \cdots].$$

   Show that the image of s is an algebraic subvariety X of $\mathbf{P}_{\mathbf{C}}^{mn-1}$, and describe all linear subspaces of $\mathbf{P}_{\mathbf{C}}^{mn-1}$ lying on X.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

October 9, 1991 (Day 2)

1. a) State the Meyer-Vietoris, Excision and Kunneth properties of the singular homology theory.

   b) Consider the trefoil knot

K in $\mathbf{R}^3 \cup \infty = S^3$

Compute $H_*(S^3 - K)$ and $H_*(S^3, S^3 - K)$.

2. Let $M = (m_{ij})$ be a symmetric $n \times n$ matrix over the field $\mathbf{F}_2$ of 2 elements. Show that the vector $(m_{11}, m_{22}, \cdots, m_{nn})$ is in the span of the rows of M.

3. Define $\mathbf{CP}^1$ in the usual way as the quotient of $\mathbf{C}^2 \setminus \{0\}$ by the action of the multiplicative group, $\mathbf{C}^*$, of nonzero, complex numbers. That is, $\lambda \in \mathbf{C}^*$ acts on $\mathbf{C}^2 \setminus \{0\}$ as the transformation

$$(z_1, z_2) \longrightarrow (\lambda z_1, \lambda z_2)$$

Let $\pi : \mathbf{C}^2 \setminus \{0\} \longrightarrow \mathbf{CP}^1$ be the projection map.

   a) Write down a non-trivial vector field in $\ker \pi_*$.

   b) Show that

$$\omega = i \left\{ \frac{dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2}{(|z_1|^2 + |z_2|^2)} - \frac{(\bar{z}_1 dz_1 + \bar{z}_2 dz_2) \wedge (z_1 d\bar{z}_1 + z_2 d\bar{z}_2)}{(|z_1|^2 + |z_2|^2)^2} \right\}$$

   is the pull back by π of a real valued 2-form on $\mathbf{CP}^1$.

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Day 2 (10/9/91)

4. Let $q \in \mathbf{C}$ obey $0 < |q| < 1$. For $\lambda \in \mathbf{C}$, consider

$$\varphi(\lambda) = \frac{1+\lambda}{1-\lambda} \cdot \prod_{n=1}^{\infty} \frac{(1+q^n\lambda)}{(1-q^n\lambda)} \prod_{n=1}^{\infty} \frac{(1+q^n\lambda^{-1})}{(1-q^n\lambda^{-1})}.$$

a) Exhibit a Laurent expansion for $\varphi(\lambda)$ which converges for $|q| < |\lambda| < 1$.

b) Exhibit one which converges for $1 < |\lambda| < |q|^{-1}$.

c) Show that $\varphi(\lambda)$ defines a meromorphic function on $\mathbf{C} \setminus \{0\}$ with poles at $\{q^n\}_{n \in \mathbf{Z}}$.

d) Prove $\varphi(q\lambda) = -\varphi(\lambda)$.

5. Let $\{x_n\}$ be a bounded sequence in a separable Hilbert space. Show that $\{x_n\}$ has a subsequence which converges weakly.

6. Consider the hypersurface

$$x^n + y^n + z^n = 0 \quad \text{in } \mathbf{CP}^2.$$

a.) Prove that it defines a nonsingular curve and compute its genus, g.

b.) Construct g linearly independent holomorphic differentials on the curve.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

October 10, 1991 (Day 3)

1. a) Define the degree of a map of one compact oriented manifold into another one. Can you map S^2 to the surface

$\Sigma =$ 

of genus two with degree 2?

Can you map Σ to S^2 with degree 2? Explain your answer.

b) Consider the map $\Sigma \longrightarrow S^2$ given by assigning to $p \in \Sigma$ the unit normal X_p to Σ at p and translating to the origin. What is its degree? Explain your answer.

2. Show that the polynomial $f(x) = x^4 - 5$ is irreducible over $\mathbf{Q}$.

   a) Calculate the degree of the splitting field K of f over $\mathbf{Q}$, and the Galois group, G, of $K/\mathbf{Q}$.

   b) Give an explicit representation of G as a subgroup of S_4, the symmetric group on 4 letters.

   c) How many fields of degree 4 over $\mathbf{Q}$ are contained in K?

3. A surface $S \subset \mathbf{R}^3$ is called *ruled* if through every point of S there passes a straight line contained in S. Show that a ruled surface has Gaussian curvature less than or equal to zero everywhere.

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Day 3 (10/10/91)

4. **Prove:**

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}dx = \frac{(1-a)\cdot\pi}{4\cos(a\pi/2)}.$$

Here, $-1 < a < 3$.

5. Let f be a smooth, complex value function on $\mathbf{R}$ which is such that

$$\lim_{|t|\to\infty} |t|^2(|f(t)| + |f'(t)|) \longrightarrow 0.$$

Also, let $\hat{f}(p) \equiv \int_{-\infty}^{\infty} dx e^{ip\cdot x} f(x)$ be f's Fourier transform (so $f = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ip\cdot x}\hat{f}(p)$).

Prove the inequality

$$\left(\int_{-\infty}^{\infty} dx\ x^2|f(x)|^2\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} \frac{dp}{2\pi} p^2|\hat{f}(p)|^2\right)^{1/2} \geq \frac{1}{2}\int_{-\infty}^{\infty} dx|f(x)|^2.$$

6. a) Let V be the vector space of polynomials of degree at most n in one variable z over a field k of characteristic 0, and let $a_1, \cdots, a_{n+1}$ be any $n+1$ distinct elements of k. Show that any polynomial $f(z) \in V$ can be written in the form

$$f(z) = c_1(z-a_1)^n + c_2(z-a_2)^n + \cdots + c_{n+1}(z-a_{n+1})^n$$

for some choice of $c_i \in k$. Show that this is false if we do not assume $\mathrm{char}(k) = 0$.

b) Assuming once more that $\mathrm{char}(k) = 0$, what is the smallest integer ℓ such that any polynomial $f(z) \in V$ can be written in the form

$$f(z) = c_1(z-a_1)^n + c_2(z-a_2)^n + \cdots + c_\ell(z-a_\ell)^n$$

for some choice of c_i *and* a_i?

No Quals

Spring 90-91

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

October 1, 1990 (Day 1)

Cliff 1. Let $a_{i,j}(t)$, $1 \le i,j \le n$ be n^2 smooth functions of a variable t, and let $r(t)$ and $D(t)$ be the rank and determinant of the matrix $(a_{i,j}(t))$. Show that the derivative

$$D'(0) = 0$$

whenever $r(0) \le n-2$, and more generally that the k^{th} derivative

$$D^{(k)}(0) = 0$$

whenever $r(0) < n-k$.

Joseph 2. Let $f(t)$ be a polynomial of degree n with rational coefficients, $\alpha_1, \ldots, \alpha_n$ – its complex roots. Suppose we know that $\alpha_3 = \alpha_1 + \alpha_2$ and that there are no other relations of this type (i.e., if $\alpha_k = \alpha_i + \alpha_j$, then $k = 3$ and $(i,j) = (1,2)$ or $(2,1)$).

a) Show that $\alpha_3 \in \mathbf{Q}$.

b) Show that α_1, α_2 lie in a quadratic extension of $\mathbf{Q}$.

Cliff 3. Let $D = \{z \in \mathbf{C} : |z| < 1\}$.

Let f be a holomorphic function on D.

a) Prove: 0 cannot be an isolated zero of $Re(f)$.

b) Let g be a complex valued, smooth function which is non-vanishing on D. Let u be a smooth real valued function on D such that $f = g \cdot u$ is holomorphic.

Prove: f is non-vanishing on D.

Oct 1 90

Smith

4. Let $\mathbf{P}^2$ be the projective plane over the complex numbers, with homogeneous coordinates X, Y, Z; and let V be the vector space of homogeneous polynomials $F(X, Y, Z)$ of degree 2. Let $p_1, \cdots, p_5$ be points in $\mathbf{P}^2$ with no three collinear, and let $W_k \subset V$ be the subspace of polynomials vanishing at the points $p_1, \cdots, p_k$. Show that W_k has dimension $6 - k$, ($k \leq 5$).

   What can the dimension of W_6 be if a sixth point, p_6, is added? (Still, no three are collinear.)

Raoul

5. Let S^2 have its standard, round metric.

   Let $\Omega_0 S^2$ denote the space of smooth maps from S^1 into S^2 which send the north pole of S^1 to the north pole of S^2. Give $\Omega_0 S^2$ the C^∞-topology. Define a map $h : \Omega_p S^2 \longrightarrow S^1 (= SO(2))$ by assigning to a curve the holonomy of the Levi-Civita connection.

   Prove that h is not homotopic to the constant map.

Stefano

6. Let G be a finite group. A G-torser on a topological space X is a covering $\pi : Y \to X$ together with a continuous action of G on Y, such that G acts simply transitively on every fiber of π. We want to describe G-torsers on X up to isomorphism.
   (a) Do it when $X = D^n$, the unit ball in $\mathbf{R}^n$.
   (b) Do it when $X = S^1$.
   (c) What is the general description if X is a finite CW-complex. Show that up to isomorphism there are only a finite number of G-torsers on X.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

October 2, 1990 (Day 2)

1. Let G be a finite group, and let C be the center of G.
   a) Show that the index, $(G : C)$, is not a prime number.
   b) Give an example where $(G : C) = 4$.

2. Let f be a continuous complex valued function on $\mathbf{R}/\mathbf{Z}$ with Fourier expansion $f(x) = \sum_{n \in \mathbf{Z}} a_n e^{2\pi i n x}$.
   a) Give an integral formula for the Fourier coefficients a_n.
   b) If f has p continuous derivatives, show that $|a_n| < \dfrac{C_f}{|n|^p}$, where C_f is a constant depending on f.

3. Let $V \cong \mathbf{C}^n$ be an n-dimensional complex vector space, and let $W = Sym^2 V^*$ be the vector space of symmetric bilinear forms on V. Let $G = GL_n(\mathbf{C})$. The action of G on V induces an action of G on W and hence on the Grassmannian $G(2, W)$ of two-dimensional spaces of quadratic forms. Show that the action of G on $G(2, W)$ has an open dense orbit in case $n = 3$, but not if $n = 4$.

Oct 2 90

Day 2, 10/2/90

4. Let $Q \subset \mathbf{C}$ be the quarter-disc $\{z | \text{Im } z \geq 0,\ \text{Re } z \geq 0,\ |z| \leq 1\}$ and α the function $\alpha(z) = i\frac{1-z}{1+z}$ for $z \in \mathbf{C}, z \neq -1$.
   a) Show that $\alpha(z) \in Q$ if and only if $z \in Q$.
   b) Let f be the conformal map from the unit disc $\{z \in \mathbf{C} : |z| \leq 1\}$ onto Q taking $1, e^{2\pi i/3}$ and $e^{4\pi i/3}$ to $0, 1, i$, respectively. Find $f(0)$.

5. Consider a curve $C \subset \mathbf{R}^3$ such that for any two points $a, b \in C$, there exists an isometry of $\mathbf{R}^3$ taking a to b and preserving C. Prove that in some coordinates, C can be parametrized as $t \to (\alpha t, \beta\cos(\gamma t), \beta\sin(\gamma t))$ for constants (α, β, γ).

6. Consider the 2 torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$.
   a) Prove that it can be covered by three contractible, open sets.
   b) Prove that it cannot be covered with two contractible, open sets.
   c) What is the minimal number of contractible, open sets for a surface of genus g?
   d) Prove that $T^3 = S^1 \times S^1 \times S^1 = \mathbf{R}^3/\mathbf{Z}^3$ is *not* covered by three contractible. open sets.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

October 3, 1990 (Day 3)

1. Let $c_n = \dfrac{(2n)!}{n!(n+1)!}$; c_n is called the n^{th} *Catalan number.*

   a. Show that c_n is an integer.

   b. Let $f(x) = \sum c_n t^n$ be the generating function associated to the sequence $c_0, c_1, \cdots$.

      Show that
$$f(t) = \frac{1-\sqrt{1-4t}}{2t}.$$

   c. Using part b), show that the Catalan numbers satisfy the recursion relation
$$c_n = \sum_{i+j=n-1} c_i c_j.$$

2. Let f be a 2-times continuously differentiable function on $[0,1]$.

   Set
$$\sigma_n \equiv \sum_{m=1}^{n} f\left(\frac{m}{n}\right).$$

   a) Prove $\sigma_n = \gamma + \frac{a_1}{n} + o(\frac{1}{n})$ where $\gamma = \int_0^1 f(t)dt$.

   b) Compute a_1 in terms of f.

   c) Prove $\sigma_n = \gamma + \frac{a_1}{n} + \frac{a_2}{n^2} + o(\frac{1}{n^2})$.

   d) Compute a_2 in terms of f.

3. Let $\mathbf{P}^5$ parametrize the projective space of symmetric 3×3 matrices modulo scalars. Let $X \subset \mathbf{P}^5$ be the hypersurface consisting of matrices of rank at most 2.
   a) Show that the singular locus $Y = \mathrm{Sing}(X)$ consists of the matrices of rank 1.
   b) Show that Y is smooth, isomorphic to $\mathbf{P}^2$, and of degree 4 (i.e., "most" planes in $\mathbf{P}^5$ of codimension 2 intersect Y in exactly 4 points).
   c) If $a, b \in Y$ and $a \neq b$, let $\overline{ab} \subset \mathbf{P}^5$ be the $\mathbf{P}^1$ spanned by a and b. Let $Z = \bigcup_{\substack{a,b \in Y \\ a \neq b}} \{\overline{ab}\}$. Compute the dimension and the degree of Z.

4. Evaluate the definite improper integral
$$\int_0^\infty \frac{\cos ax}{\cosh bx} dx$$
where $b > 0$ and a is an arbitrary real number.
[$\cosh t = \cos it = \frac{1}{2}(e^t + e^{-t})$ is the hyperbolic cosine of t.]

5. Let X be a 2×2, nonzero, real matrix with zero trace.
   a) Give a quadratic condition on X for e^{tX} to be a non-compact, 1-parameter subgroup of $SL(2, \mathbf{R})$.
   b) Describe all the possible conjugacy classes of 1-parameter subgroups in $SL(2, \mathbf{R})$.
   c) Prove that the matrix
$$\begin{pmatrix} -2 & 0 \\ 0 & -1/2 \end{pmatrix} \in SL(2, \mathbf{R})$$
   is on *no* 1-parameter subgroup of $SL(2, \mathbf{R})$.
   d) Can you construct a point with this property on $SO(3)$? Explain.

please go to page 3

Day 3, 10/3/90

6. a) Draw the universal cover $\widetilde{X}$ of the space

$$X = S^2 \cup C$$

where C is the chord joining the North to the South Pole. Thus

$X =$ 

b) What is the relation of $\pi_2(X)$ and $\pi_2(\widetilde{X})$?

c) Compute $\pi_2(X)$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

February 6, 1990 (Day 1)

1. Let f and g be entire holomorphic functions satisfying the identity

$$f(z)^2 = g(z)^6 - 1$$

for all $z \in \mathbf{C}$. Show that f and g are constant. Would the same conclusion be valid if f and g were assumed to be entire meromorphic functions?

Hint: Think about the algebraic curve $y^2 = x^6 - 1$.

2. a) State the fundamental classification theorem about finitely generated $\mathbf{Z}$-modules.

   b) Let $\varphi : \mathbf{Z}^3 \longrightarrow \mathbf{Z}^3$ be defined by the matrix

$$\begin{pmatrix} -12 & 6 & 0 \\ 58 & 34 & 18 \\ 18 & 12 & 6 \end{pmatrix}.$$

   Decompose $\ker(\varphi)$ and $\operatorname{coker}(\varphi)$ into direct sums of cyclic groups.

3. a) Compute the spectrum of $\dfrac{d^2}{dt^2}$ with Dirichlet boundary conditions on $[0, \pi]$.

   b) Construct the Dirichlet inverse, G, of $\dfrac{d^2}{dt^2}$ on $[0, \pi]$.

   c) Let L^2 denote the completion of the space of C^∞ functions on $[0, \pi]$ using the norm

$$\|f\|_2 = \int_0^\pi ds |f(s)|^2.$$

   Prove that G extends to L^2 as a *compact* operator from L^2 to itself.

   d) Let V be a smooth function on $[0, \pi]$ with $|V| < 1$. Prove that

$$\frac{d^2}{dt^2} + V(t)$$

   has a Dirichlet inverse on $[0, \pi]$ which extends to L^2 as a bounded operator.

4. Let $C \subset \mathbb{C}P^N$ be a nonsingular complex curve of some genus, g.

Let $p_1, \cdots, p_k$ be distinct points on C, and let $n_1, \cdots, n_k$ be positive integers.

1) Estimate an upper bound on the dimension of the vector space of meromorphic functions on C which have poles of order $\leq n_j$ at p_j and are regular elsewhere.
2) Do the same for the vector space of meromorphic differentials on C which have poles of order $\leq n_j$ at p_j and are regular elsewhere.

5. Let X be the figure eight:

a). How many connected, 3-sheeted covering spaces of X are there, up to isomorphism over X? Draw them.

b). How many of these are normal (i.e., Galois) covering spaces?

6. Produce a curve in $\mathbf{R}^3$ with given constant curvature, $\kappa \neq 0$, and torsion, $\tau \neq 0$.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

February 7, 1990 (Day 2)

1. Let f be a holomorphic function on a domain containing the closed disc $\{z : |z| \leq 3\}$, such that

$$f(1) = f(i) = f(-1) = f(-i) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|.$$

2. Let R be a commutative ring.

Suppose a and b are elements of R such that the ideal $Ra + Rb$ is principal. Prove that the ideal $Ra \cap Rb$ is principal.

3. A lighthouse situated at the origin casts a very narrow beam of light that rotates there one full turn every second. How fast is the image of the light beam moving along the ellipse given by the equation

$$x^2 + 3y^2 = 6$$

at the time when the beam makes an angle of 45° with the positive x-axis? Here, (x, y) have units of meters, and 6 is 6 (meters)^2. Your answer should be in meters/second.

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4. Let $V = \mathbf{C}^{mn}$ be the vector space of complex $m \times n$ matrices, $\mathbf{P}V = \mathbf{P}^{mn-1}$ the associated projective space, and let $M_k \subset \mathbf{P}V$ be the subset of matrices of rank k or less.

a). Show that M_k is an algebraic subvariety of $\mathbf{P}V$.

b). Find the dimension of M_k.

c). Show that M_k is irreducible.

5. Let S^2 be the 2-sphere, $T = S^1 \times S^1$ the torus, and Z the compact orientable surface of genus 2.

a). Does there exist a continuous map $f : S^2 \longrightarrow T$ not homotopic to a constant?

b). Does there exist a continuous map $f : S^2 \longrightarrow Z$ not homotopic to a constant?

c). Does there exist a continuous map $f : T \longrightarrow Z$ not homotopic to a constant?

Justify your answers.

6. k is a real number.

Consider the 3-vector fields

$$X = (z + k^2 x)\frac{\partial}{\partial x} - (x + ky)\frac{\partial}{\partial y}$$

$$Y = y\frac{\partial}{\partial y} - (z + k^2 x)\frac{\partial}{\partial z}$$

$$Z = y\frac{\partial}{\partial x} - (x + ky)\frac{\partial}{\partial z}.$$

a) Prove: On the complement of $0 \in \mathbf{R}^3$, $\{X, Y, Z\}$ span a subbundle, $E \subset T(\mathbf{R}^3 \setminus \{0\})$.

b) For what value of k is E an integrable subbundle?

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

February 8, 1990 (Day 3)

1. Compute:

$$\int_0^\infty \frac{(\log\ x)^2 dx}{(1+x^2)}.$$

2. Let G be a finite group, all of whose Sylow subgroups are normal. Prove: G is the product of its Sylow subgroups.

3. Let L^2 denote the completion of the space of smooth functions of compact support on $[0,\infty)$, C_0^∞, using the norm

$$\|f\|_2^2 = \int_0^\infty |f(t)|^2 dt.$$

Let L_1^2 denote the completion of the same space, C_0^∞, using the norm

$$\|f\|_{2,1} = (\|f\|_2^2 + \|\frac{\partial f}{\partial t}\|_2^2)^{1/2}.$$

a) Prove that $\frac{\partial}{\partial t} : C_0^\infty \longrightarrow C_0^\infty$ extends to define a bounded operator from L_1^2 to L^2.

b) Prove $\dim(\mathrm{coker}\ \frac{\partial}{\partial t} : L_1^2 \longrightarrow L^2) = \infty$.

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4. Consider the polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Let Δ be the discriminant of f, that is,

$$\Delta = \prod_{i<j} (\mu_i - \mu_j)^2$$

where $\mu_1, \cdots, \mu_n$ are the roots of f. Δ may be viewed as a function of the coefficients $a_0, \cdots, a_{n-1}$.

a). Show that Δ is a polynomial in $a_0, \cdots, a_{n-1}$.

b). Find the degree of Δ as a polynomial in the a_i.

5. Suppose X is a smooth manifold with an open cover by $N < \infty$ sets $\{B_n\}_{n=1}^{N}$, where each B_n is contractible. Assume that $\pi_0(B_n \cap B_m) \leq k$ for all n and m. Give an upper bound to the first Betti number of X.

6. Let $\Sigma \hookrightarrow \mathbf{R}^3$ be an embedded, closed surface of genus 3.

Define the Gauss map, $\varphi : \Sigma \longrightarrow S^2$ by associating to each point $x \in \Sigma$, the outward pointing, unit normal vector.

Let ω denote the volume form (total volume $= 4\pi$) on S^2.

Compute $\int_\Sigma \varphi^* \omega$.