Solutions of Qualifying Exams I, 2014 Spring

1. (ALGEBRA) Let $k = \mathbb{F}_q$ be a finite field with q elements. Count the number of monic irreducible polynomials of degree 12 over k.

Solution. Let $G := \mathrm{Gal}(\mathbb{F}_{q^{12}}/\mathbb{F}_q)$ act naturally on $\mathbb{F}_{q^{12}}$. The set of monic irreducible polynomials of degree 12 are in one-to-one correspondence with the set of G-orbits of order 12 in $\mathbb{F}_{q^{12}}$. An orbit $G\alpha$ has order 12 exactly when the subfield $\mathbb{F}_q(\alpha)$ coincides with $\mathbb{F}_{q^{12}}$, i.e., exactly when

$$\alpha \in \mathbb{F}_{q^{12}} \setminus \bigcup_{\mathbb{F}_q \le K \subsetneq \mathbb{F}_{q^{12}}} K$$

The maximal proper subfields of $\mathbb{F}_{q^{12}}$ are $\mathbb{F}_{q^6}$ and $\mathbb{F}_{q^4}$. By inclusion-exclusion principle, the number of the polynomials sought is equal to

$$\frac{q^{12} - q^6 - q^4 + q^2}{12}.$$

2. (ALGEBRAIC GEOMETRY) (a) Show that the set of lines $L \subset \mathbb{P}^3_{\mathbb{C}}$ may be identified with a quadric hypersurface in $\mathbb{P}^5_{\mathbb{C}}$.

(b) Let $L_0 \subset \mathbb{P}^3_{\mathbb{C}}$ be a given line. Show that the set of lines not meeting L_0 is isomorphic to the affine space $\mathbb{A}^4_{\mathbb{C}}$.

Solution. (a) If $\mathbb{P}^3 = \mathbb{P}V$ is the projective space of one-dimensional subspaces of a 4-dimensional vector space V, then we associate to the line L spanned by two vectors $v, w \in V$ the wedge product $v \wedge w \in \mathbb{P}\bigwedge^2 V \cong \mathbb{P}^5$. Since a 2-form $\eta \in \bigwedge^2 V$ is decomposable if and only if $Q(\eta) = \eta \wedge \eta = 0 \in \bigwedge^4 V \cong \mathbb{C}$, this identifies the set of lines with the zeroes of the quadratic form Q.

(b) Choose 2 planes $\Lambda, \Lambda' \subset \mathbb{P}^3$ containing L_0. Any line not meeting L_0 is determined by its points of intersection with the two planes, giving an isomorphism between the set of lines not meeting L_0 and

$$(\Lambda \setminus L_0) \times (\Lambda' \setminus L_0) \cong \mathbb{A}^2 \times \mathbb{A}^2 \cong \mathbb{A}^4.$$

3. (COMPLEX ANALYSIS) (a) Compute

$$\int_{|z|=1} \frac{z^{31}}{(2\bar{z}^{\,2} + 3)^2 \, (\bar{z}^{\,4} + 2)^3} \, dz$$

Note that the integrand is not a meromorphic function.

(b) Evaluate the integral

$$\int_{x=0}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx$$

by using the theory of residues. Justify carefully all the limiting processes in your computation.

Solution. (a) Since $\bar{z} = \frac{1}{z}$ for $|z| = 1$, it follows that

$$\int_{|z|=1} \frac{z^{31}}{\left(2\bar{z}^{\,2}+3\right)^2 \left(\bar{z}^{\,4}+2\right)^3}\, dz$$
$$= \int_{|z|=1} \frac{z^{31}}{\left(2\left(\frac{1}{z}\right)^2+3\right)^2 \left(\left(\frac{1}{z}\right)^4+2\right)^3}\, dz.$$

Use the change of variables $z = \frac{1}{w}$ to transform the integral to

$$-\int_{|w|=1} \frac{\frac{1}{w^{31}}}{\left(2w^2+3\right)^2 \left(w^4+2\right)^3} \left(-\frac{dw}{w^2}\right).$$

The negative sign in front of the integral comes from the change of orientation when the parametrization $z = e^{i\theta}$ for $0 \le \theta \le 2\pi$ is transformed to the parametrization $w = e^{-i\theta}$ for $0 \le \theta \le 2\pi$. This new integral can be rewritten as

$$\int_{|w|=1} \frac{dw}{w^{33}\left(3+2w^2\right)^2 \left(2+w^4\right)^3},$$

which is equal to $2\pi i$ times the residue of the meromorphic function

$$\frac{1}{w^{33}\left(3+2w^2\right)^2 \left(2+w^4\right)^3}$$

at $w = 0$. We have the power series expansion of the factor

$$\frac{1}{\left(3+2w^2\right)^2} = \frac{1}{9}\frac{1}{\left(1+\frac{2}{3}w^2\right)^2}$$
$$= \frac{1}{9}\sum_{k=0}^{\infty} \frac{(-2)(-3)\cdots(-2-k+1)}{k!}\left(\frac{2}{3}w^2\right)^k$$

at $w=0$ and the power series expansion of the factor

$$\begin{aligned}\frac{1}{(2+w^4)^3} &= \frac{1}{8}\frac{1}{\left(1+\frac{1}{2}w^4\right)^3} \\ &= \frac{1}{8}\sum_{\ell=0}^{\infty}\frac{(-3)(-4)\cdots(-3-\ell+1)}{\ell!}\left(\frac{1}{2}w^4\right)^{\ell}\end{aligned}$$

at $w=0$. Contributions to the residue in question from the two power series expansions come from $2k+4\ell=32$, which means that k must be divisible by 2 and there are only 9 choices for ℓ from 0 to 8 inclusively (with the corresponding value $k=\frac{32-4\ell}{2}=16-2\ell$). Hence the residue in question is equal to the following sum

$$\frac{1}{72}\sum_{\ell=0}^{8}\frac{(-2)(-3)\cdots(-2-(16-2\ell)+1)}{(16-2\ell)!}\left(\frac{2}{3}\right)^{16-2\ell}\frac{(-3)(-4)\cdots(-3-\ell+1)}{\ell!}\left(\frac{1}{2}\right)^{\ell}$$

of 9 terms. The final answer is that

$$\int_{|z|=1}\frac{z^{31}}{(2\bar{z}^{\,2}+3)^2\,(\bar{z}^{\,4}+2)^3}\,dz$$

is equal to

$$\frac{2\pi i}{72}\sum_{\ell=0}^{8}\frac{(-2)(-3)\cdots(-2-(16-2\ell)+1)}{(16-2\ell)!}\left(\frac{2}{3}\right)^{16-2\ell}\frac{(-3)(-4)\cdots(-3-\ell+1)}{\ell!}\left(\frac{1}{2}\right)^{\ell}.$$

(b) By Euler's formula we have $\sin x=\frac{e^{ix}-e^{-ix}}{2i}$ and

$$\begin{aligned}\sin^3 x &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3 \\ &= \frac{1}{-8i}\left(e^{3ix}-3e^{ix}+3e^{-ix}-e^{-3ix}\right) \\ &= \frac{1}{4}\left(3\,\frac{e^{ix}-e^{-ix}}{2i}-\frac{e^{3ix}-e^{-3ix}}{2i}\right).\end{aligned}$$

Thus $\sin^3 x$ is the imaginary part of

$$\frac{3}{4}\,e^{ix}-\frac{1}{4}\,e^{3ix}.$$

The power series expansion of

$$\frac{3}{4}\, e^{iz} - \frac{1}{4}\, e^{3iz}$$

is

$$\frac{3}{4}\left(1 + iz + O\left(z^2\right)\right) - \frac{1}{4}\left(1 + 3iz + O\left(z^2\right)\right) = \frac{1}{2} + O\left(z^2\right).$$

The $\mathbb{R}$-linear combination

$$\frac{3}{4}\, e^{iz} - \frac{1}{4}\, e^{3iz} - \frac{1}{2}$$

vanishes to order 2 at $z = 0$ and its imaginary part for $z = x$ real is equal to $\sin^3 x$. Let

$$f(z) = \frac{-\frac{1}{4}\, e^{3iz} + \frac{3}{4}\, e^{iz} - \frac{1}{2}}{z^3}.$$

Its behavior near $z = 0$ is given by

$$f(z) = \frac{-\frac{1}{4}\frac{(3iz)^2}{2} + \frac{3}{4}\frac{(iz)^2}{2} + O\left(z^3\right)}{z^3} = \frac{3}{4}\frac{1}{z} + O\left(z^3\right)$$

and we have a simple pole for f at $z = 0$ whose residue $\mathrm{Res}_0 f$ is $\frac{3}{4}$. Integrating

$$f(z)\, dz$$

over the boundary of the set which is equal to the upper half-disk of radius $R > 0$ minus the upper half-disk of radius r with $0 < r < R$ and letting $R \to \infty$ and $r \to 0$, we get

$$\int_{x=-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx = \mathrm{Im}\left(\pi i\, \mathrm{Res}_0 f\right) = \mathrm{Im}\left(\pi i\, \frac{3}{4}\right) = \frac{3\pi}{4}$$

and

$$\int_{x=0}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{8}.$$

To justify the limiting process, we have to show that the integral

$$\int_{C_R} f(z)\, dz$$

over the upper half-circle of radius R centered at the origin 0 approaches 0 as $R \to \infty$. This is a consequence of the fact that both $|e^{3iz}|$ and $|e^{iz}|$ are ≤ 1 for $\operatorname{Im} z \geq 0$ so that

$$\left| \int_{C_R} f(z)\, dz \right| \leq \frac{1}{R^3} \pi R \to 0 \quad \text{as} \quad R \to \infty.$$

We need also to compute the integral

$$\int_{C_r} f(z)\, dz$$

over the upper half-circle of radius r centered at the origin 0 in the counter-clockwise sense as $r \to 0+$. This is done by using

$$f(z) = \frac{3}{4}\frac{1}{z} + O\left(z^3\right)$$

and the parametrization $z = re^{i\theta}$ for $0 \leq \theta \leq \pi$ so that

$$\begin{aligned} \lim_{r \to 0+} \int_{C_r} f(z)\, dz &= \lim_{r \to 0+} \int_{C_r} \frac{3}{4}\frac{1}{z}\, dz \\ &= \int_{\theta=0}^{\pi} \frac{3}{4}\frac{1}{re^{i\theta}}\, ire^{i\theta} d\theta = \pi i\, \frac{3}{4}. \end{aligned}$$

4. (Algebraic Topology) Suppose that X is a finite connected CW complex such that $\pi_1(X)$ is finite and nontrivial. Prove that the universal covering $\tilde{X}$ of X cannot be contractible. (*Hint:* Lefschetz fixed point theorem.)

Solution. Since X is a finite CW complex, $\tilde{X}$ is also a finite CW complex. Suppose $\tilde{X}$ is contractible. Then $\tilde{X}$ has the same homology as a point, i.e. $H_0(\tilde{X}) = \mathbb{Z}$ and $H_i(\tilde{X}) = 0$ for $i \neq 0$. Then by the Lefschetz fixed point theorem any continuous map $f\colon \tilde{X} \to \tilde{X}$ has a fixed point. On the other hand, the group of covering transformations of $\tilde{X}$ is isomorphic to $\pi_1(X)$, hence is nontrivial. Since a non-identity covering transformation does not have fixed points, we obtain a contradiction. Thus $\tilde{X}$ cannot be contractible.

5. (Differential Geometry) Let $\mathbb{P}^2 = (\mathbb{C}^3 - \{0\})/\mathbb{C}^\times$, which is called the complex projective plane.

1. Show that $\mathbb{P}^2$ is a complex manifold by writing down its local coordinate charts and transitions.

2. Define $L \subset \mathbb{P}^2 \times \mathbb{C}^3$ to be the subset containing elements of the form $([x], \lambda x)$, where $x \in \mathbb{C}^3 - \{0\}$ and $\lambda \in \mathbb{C}$. Show that L is the total space of a holomorphic line bundle over $\mathbb{P}^2$ by writing down its local trivializations and transitions. It is called the tautological line bundle.

3. Using the standard Hermitian metric on $\mathbb{C}^3$ or otherwise, construct a Hermitian metric on the tautological line bundle. Express the metric in terms of local trivializations.

Sketched Solution.

1. The charts are $\phi_0 : U_0 = \{[x, y, z] : z \neq 0\} \to \mathbb{C}^2 = V_0$ by $[x, y, z] \mapsto (x/z, y/z)$, and ϕ_1, ϕ_2 are defined similarly. The transition from V_0 to V_1 is $(X, Y) \mapsto [X, Y, 1] \mapsto (1, Y/X, 1/X)$ for $X \neq 0$, and other transitions are computed in a similar way.

2. The local trivialization over U_0 is $([x, y, 1], \lambda(x, y, 1)) \mapsto ([x, y, 1], \lambda)$, and that over U_1 and U_2 are defined in a similar way. The transition over $U_0 \cap U_1$ is

   $$([x, y, 1], \lambda) \mapsto ([x, y, 1], \lambda(x, y, 1)) = ([1, y/x, 1/x], \lambda x(1, y/x, 1/x)) \mapsto ([x, y, 1], x\lambda).$$

   The transition over U_{12} and U_{02} are similarly defined.

3. Define a metric by $([x], \lambda x) \mapsto \|\lambda x\|$. Over U_0, it is given by $([x, y, 1], \lambda) \mapsto \|\lambda(x, y, 1)\|$. It is similar for the other trivializations U_1, U_2.

6. (Real Analysis) *(Schwartz's Theorem on Perturbation of Surjective Maps by Compact Maps Between Hilbert Spaces).* Let E, F be Hilbert spaces over $\mathbb{C}$, $S : E \to F$ be a compact $\mathbb{C}$-linear map, and $T : E \to F$ be a continuous surjective $\mathbb{C}$-linear map. Prove that the cokernel of $S+T : E \to F$ is finite-dimensional and the image of $S + T : E \to F$ is a closed subspace of F.

Here the compactness of the $\mathbb{C}$-linear map $S : E \to F$ means that for any sequence $\{x_n\}_{n=1}^{\infty}$ in E with $\|x_n\|_E \leq 1$ for all $n \in \mathbb{N}$ there exists some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $S(x_{n_k})$ converges in F to some element of F as $k \to \infty$.

Hint: Verify first that the conclusion is equivalent to the following equivalent statement for the adjoints $T^*, S^* : F \to E$ of T, S. The kernel of $T^* + S^*$ is finite-dimensional and the image of $T^* + S^*$ is closed. Then prove the equivalent statement.

Solution. We prove first the equivalent statement for the adjoints $T^*, S^* : F \to E$ for T, S and then at the end obtain from it the original statement for $T, S : E \to F$.

The adjoint S^* of the compact operator S is again compact (see *e.g.,* p.189 of Stein and Shakarchi's *Real Analysis*). Since T is surjective, by the open mapping theorem for Banach spaces and in particular for Hilbert spaces, the map $T : E \to F$ is open. It implies that F is the quotient of E by the kernel of T. Thus T^* is the isometry between F and the orthogonal complement of the kernel of T in E, when a Hilbert space is naturally identified with its dual by using its inner product according to the Riesz representation theorem (see *e.g.,* Theorem 5.3 on p.182 of Stein and Shakarchi's *Real Analysis*).

Now we verify that the kernel of $T^* + S^*$ is finite-dimensional by showing that its closed unit ball is compact. Take a sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in the kernel of $T^* + S^*$ with $\|y_n\|_F \leq 1$ for $n \in \mathbb{N}$. Then $T^*y_n + S^*y_n = 0$ for $n \in \mathbb{N}$. Since S^* is compact, there exists a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ such that $S^*y_{n_k} \to z$ in E for some $z \in E$. From $T^*y_{n_k} \to -z$ as $k \to \infty$ and the fact that T^* is the isometry between F and the orthogonal complement of kernel of T in E, it follows that y_{n_k} converges to the unique element $\hat{z}$ in F such that $T^*\hat{z} = -z$. Since $z = \lim_{k\to\infty} S^*y_{n_k} = S^*\hat{z}$ and $-z = T^*\hat{z}$, it follows that $T^*\hat{z} + S^*\hat{z} = 0$ and z is in the kernel of $T^* + S^*$. Thus the closed unit ball of the kernel of $T^* + S^*$ is compact. Since every locally compact Hilbert space is finite dimensional, it follows that the kernel of $T^* + S^*$ is finite-dimensional.

Now we verify that the image of $T^* + S^*$ is closed. Suppose for some sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in F we have the convergence of $T^*y_n + S^*y_n$ in E to some element z in E. We have to show that z belongs to the image of $T^* + S^*$. By replacing y_n by its projection onto the orthogonal complement $(\operatorname{Ker}(T^* + S^*))^\perp$ of the kernel of $T^* + S^*$ in F, we can assume without loss of generality that each y_n belongs to $(\operatorname{Ker}(T^* + S^*))^\perp$.

We claim that the sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in $(\operatorname{Ker}(T^* + S^*))^\perp$ is bounded in the norm $\|\cdot\|_F$ of F, otherwise we can define $\hat{y}_n = \frac{y_n}{\|y_n\|_F}$ so that $T^*\hat{y}_n + S^*\hat{y}_n \to 0$ as $n \to \infty$ with $\|\hat{y}_n\|_F = 1$ for all $n \in \mathbb{N}$. Since S^* is compact, there is a subsequence $\{\hat{y}_{n_k}\}_{k\in\mathbb{N}}$ of $\{\hat{y}_n\}_{n\in\mathbb{N}}$ with $S^*\hat{y}_{n_k}$ converging to some element u in E. Thus

$$T^*\hat{y}_{n_k} = (T^*\hat{y}_{n_k} + S^*\hat{y}_{n_k}) - S^*\hat{y}_{n_k}$$

converges to the element $-u$ in E. Since T^* is the isometry between F and the orthogonal complement of kernel of T in E, it follows that $\hat{y}_{n_k}$ converges to the unique element v in F such that $T^*v = -u$. This means that $(T^* + S^*)(v) = 0$ and $v \in \operatorname{Ker}(T^* + S^*)$. On the other hand, v being the limit of the sequence $\hat{y}_{n_k}$ in $(\operatorname{Ker}(T^* + S^*))^\perp$ must be in $(\operatorname{Ker}(T^* + S^*))^\perp$ also. Thus, $v = 0$, which contradicts the fact that it is the limit of $\hat{y}_{n_k}$ with $\|\hat{y}_{n_k}\|_F = 1$ for all $k \in \mathbb{N}$. This finishes the proof of the claim that sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in $(\operatorname{Ker}(T^* + S^*))^\perp$.

Since the sequence of points $\{y_n\}_{n\in\mathbb{N}}$ in $(\operatorname{Ker}(T^* + S^*))^\perp$ is bounded in the norm $\|\cdot\|_F$ of F, by the compactness of S^* we can select a a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ with $S^*y_{n_k}$ converging to some element w in E. Thus

$$T^*y_{n_k} = (T^*y_{n_k} + S^*y_{n_k}) - S^*y_{n_k}$$

converges to the element $z-w$ in E. Since T^* is the isometry between F and the orthogonal complement of kernel of T in E, it follows that y_{n_k} converges to the unique element t in F such that $T^*t = z-w$. With $w = S^*t = \lim_{k\to\infty} y_{n_k}$, this implies that $(T^* + S^*)(t) = z$. This finishes the proof that the image of $T^* + S^*$ is closed.

Since we now know that the image of T^*+S^* is closed, it follows from the Riesz representation theorem that the map $S+T$ maps E onto the orthogonal complement $\operatorname{Ker}(T^*+S^*)^\perp$ of the kernel $\operatorname{Ker}(T^*+S^*)$ of T^*+S^* in F. Hence the image of $T+S$ is closed and the cokernel of $T+S$ is finite-dimensional.

Solutions of Qualifying Exams II, 2014 Spring

1. (Algebra) Let A be a finite group of order n, and let $V_1, \cdots, V_k$ be its irreducible representations.

(a) Show that the dimensions of the vector spaces V_i satisfy the equality $\sum_{i=1}^{k}(\dim V_i)^2 = n$.

(b) What are the dimensions of the irreducible representations of the symmetric group S_6 of six elements.

Solution. (a) Use the character theory and show that V_i appears $(\dim V_i)$ times in the regular representation $\mathbb{C}[A]$.

(b) Irreducible representations of S_6 correspond to conjugacy classes in S_6, and then to partitions of 6, of which there are $p(6) = 11$. Then use the "hook-length formula",

$$\dim V_\lambda = \frac{d!}{\prod(\text{ hook lengths })}.$$

They are: 16, 10 (twice), 9 (twice), 5 (four times) and 1 (twice).

2. (Algebraic Geometry) Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree ≥ 3.

(a) Show that C admits a regular map $f : C \to \mathbb{P}^1$ of degree $d - 1$.

(b) Show that C does not admit a regular map $f : C \to \mathbb{P}^1$ of degree e with $0 < e < d - 1$.

Solution. (a) Solution: Simply project from any point $p \in C$ to a complementary line.

(b) Since the canonical series of C is cut on C by plane curves of degree $d-3$, by Riemann-Roch the general fiber of any map $f : C \to \mathbb{P}^1$ of degree e must consist of e points of C that fail to impose independent conditions on curves of degree $d-3$. But any set $d-2$ or fewer points in the plane impose independent conditions on curves of degree $d - 3$.

3. (Complex Analysis) Suppose that f is holomorphic in an open set containing the closed unit disk $\{|z| \leq 1\}$ in $\mathbb{C}$, except for a pole at z_0 on the unit circle $\{|z| = 1\}$. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denote the power series expansion of f in the open unit disk $\{|z| < 1\}$, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Solution. Since z_0 is the only pole of the meromorphic function f on an open set containing the closed unit disk in $\mathbb{C}$, we can express $f(z)$ in the form

$$\sum_{k=1}^{m} \frac{A_k}{(z - z_0)^k} + g(z)$$

with $A_1, \cdots, A_m \in \mathbb{C}$, where $m \geq 1$ and $A_m = h(z_0) \neq 0$ and $g(z)$ is a power series $\sum_{n=0}^{\infty} b_n z^n$ with radius of convergence $R > 1$. For any positive number r with $|z_0| < r < R$ we can find a positive number B such that

$$|b_n| \leq \frac{B}{r^n}$$

for all nonnegative integer n. By using the binomial expansion of $\frac{1}{(z-z_0)^k}$ (or differentiating the geometric series $\frac{1}{z-z_0}$ in z $(k-1)$-times) and noting that $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$, we have

$$a_n = b_n + \sum_{k=1}^{m} (-1)^k A_k \frac{(n+k-1)(n+k-2)\cdots(n+2)(n+1)}{(k-1)!\,(z_0)^{n+k}}.$$

In the computation of the limit

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}},$$

since $A_m = h(z_0) \neq 0$ and $\frac{1}{r} < \left|\frac{1}{z_0}\right|$ and $|b_n| \leq \frac{B}{r^n}$, the dominant term from a_n is

$$(-1)^m A_m \frac{(n+m-1)(n+m-2)\cdots(n+2)(n+1)}{(m-1)!\,(z_0)^{n+m}}$$

and we get

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\frac{(-1)^m A_m \frac{(n+m-1)(n+m-2)\cdots(n+2)(n+1)}{(m-1)!(z_0)^{n+m}}}{(-1)^m A_m \frac{(n+m)(n+m-1)\cdots(n+3)(n+2)}{(m-1)!(z_0)^{n+1+m}}}=z_0.$$

The dominant term from a_n means that a_n minus the dominant term and then divided by the dominant term would have limit zero when $n\to\infty$.

4. (Algebraic Topology) Show that if $n>1$, then every map from the real projective space $\mathbb{RP}^n$ to the n-torus T^n is null-homotopic.

Solution. Recall that $\pi_1(\mathbb{RP}^n)=\mathbb{Z}/2\mathbb{Z}$ and

$$\pi_1(T^n)=\pi_1(S^1\times\cdots\times S^1)=\mathbb{Z}^n.$$

Now if $f\colon\mathbb{RP}^n\to T^n$ is any map, then the induced homomorphism

$$f_*\colon\pi_1(\mathbb{RP}^n)\to\pi_1(T^n)$$

must be trivial because $\mathbb{Z}^n$ has no nontrivial elements of finite order. Let $p\colon\mathbb{R}^n\to T^n$ be the standard covering map. Then, by the general lifting lemma, we obtain a continuous map $\tilde{f}\colon\mathbb{RP}^n\to\mathbb{R}^n$ such that $f=p\circ\tilde{f}$. Since $\mathbb{R}^n$ is contractible, we obtain that $\tilde{f}$ is nullhomotopic, from which it follows that f is nullhomotopic.

5. (Differential Geometry) Let $\mathbb{S}^2:=\{x^2+y^2+z^2=1\}\subset\mathbb{R}^3$ be the unit sphere in the Euclidean space. Let $C=\{(r\cos t,r\sin t,h):t\in\mathbb{R}\}$ be a circle in $\mathbb{S}^2$, where $r,h>0$ are constants with $r^2+h^2=1$.

1. Compute the holonomy of the sphere $\mathbb{S}^2$ (with the standard induced metric) around the circle C.
2. By using Gauss-Bonnet theorem or otherwise, compute the total curvature

$$\int_D\kappa\,\mathrm{d}A$$

where $D=\mathbb{S}^2\cap\{z\geq h\}$ is the disc bounded by the circle C, and $\mathrm{d}A$ is the area form of $\mathbb{S}^2$.

Sketched Solution.

1. The holonomy is rotation by $2\pi h$.
2. The total curvature is $2\pi-2\pi\sqrt{r^2h^2+(1-r^2)^2}$.

6. (Real Analysis) *(Commutation of Differentiation and Summation of Integrals).* Let Ω be an open subset of $\mathbb{R}^d$ and $a < b$ be real numbers. For any positive integer n let $f_n(x, y)$ be a complex-valued measurable function on $\Omega \times (a, b)$. Let $a < c < b$. Assume that the following three conditions are satisfied.

(i) For each n and for almost all $x \in \Omega$ the function $f_n(x, y)$ as a function of y is absolutely continuous in y for $y \in (a, b)$.

(ii) The function $\frac{\partial}{\partial y} f_n(x, y)$ is measurable on $\Omega \times (a, b)$ for each n and the function

$$\sum_{n=1}^{\infty} \left| \frac{\partial}{\partial y} f_n(x, y) \right|$$

is integrable on $\Omega \times (a, b)$.

(iii) The function $f_n(x, c)$ is measurable on Ω for each n and the function $\sum_{n=1}^{\infty} |f_n(x, c)|$ is integrable on Ω.

Prove that the function

$$y \mapsto \int_{x\in\Omega} \sum_{n=1}^{\infty} f_n(x, y) dx$$

is a well-defined function for almost all points y of (a, b) and that

$$\frac{d}{dy} \int_{x\in\Omega} \sum_{n=1}^{\infty} f_n(x, y) dx = \sum_{n=1}^{\infty} \int_{x\in\Omega} \left(\frac{\partial}{\partial y} f_n(x, y) \right) dx$$

for almost all $y \in (a, b)$.

Hint: Use Fubini's theorem to exchange the order of integration and use convergence theorems for integrals of sequences of functions to exchange the order of summation and integration.

Solution. The theorem of Fubini which we will use states that if $F(x, y)$ on $\Omega_1 \times \Omega_2$ (with Ω_j open in $\mathbb{R}^{d_j}$ for $j = 1, 2$) and if

$$\int_{(x,y)\in\Omega_1\times\Omega_2} |F(x, y)| < \infty,$$

then

$$\int_{x\in\Omega_1}\left(\int_{y\in\Omega_2} F(x,y)dy\right)dx = \int_{y\in\Omega_2}\left(\int_{x\in\Omega_1} F(x,y)dx\right)dy.$$

One consequence of the theorem of dominated convergence which we will use is the folloing exchange of integration and summation. If $F_n(x)$ is a sequence of measurable functions on an open subset $\tilde{\Omega}$ of $\mathbb{R}^{\tilde{d}}$ such that

$$\int_{x\in\tilde{\Omega}}\sum_{n=1}^{\infty}|F_n(x)| < \infty,$$

then

$$\int_{x\in\tilde{\Omega}}\sum_{n=1}^{\infty}F_n(x) = \sum_{n=1}^{\infty}\int_{x\in\tilde{\Omega}}F_n(x).$$

These two results make it possible for us to both exchange the order of integration and the order of summation and integration in the following equation for $a < \eta < b$,

(†)

$$\int_{y=c}^{\eta}\left(\sum_{n=1}^{\infty}\int_{x\in\Omega}\left(\frac{\partial}{\partial y}f_n(x,y)\right)dx\right)dy = \int_{x\in\Omega}\left(\sum_{n=1}^{\infty}\int_{y=c}^{\eta}\left(\frac{\partial}{\partial y}f_n(x,y)\right)dy\right)dx,$$

because the function

$$\sum_{n=1}^{\infty}\left|\frac{\partial}{\partial y}f_n(x,y)\right|$$

is integrable on $\Omega\times(a,b)$. Since for almost all $x\in\Omega$ the function $f_n(x,y)$ as a function of y is absolutely continuous in y, it follows that

$$\int_{y=c}^{\eta}\left(\frac{\partial}{\partial y}f_n(x,y)\right)dy = f_n(x,\eta) - f_n(x,c)$$

for almost all $x\in\Omega$, which implies that

$$\begin{aligned}\int_{x\in\Omega}\left(\sum_{n=1}^{\infty}\int_{y=c}^{\eta}\left(\frac{\partial}{\partial y}f_n(x,y)\right)dy\right)dx &= \int_{x\in\Omega}\left(\sum_{n=1}^{\infty}(f_n(x,\eta)-f_n(x,c))\right)dx\\ &= \int_{x\in\Omega}\left(\sum_{n=1}^{\infty}f_n(x,\eta)\right)dx - \int_{x\in\Omega}\left(\sum_{n=1}^{\infty}f_n(x,c)\right)dx,\end{aligned}$$

because $\sum_{n=1}^{\infty} |f_n(x,c)|$ is integrable on Ω. Putting this together with (†) yields
(‡)

$$\int_{y=c}^{\eta} \left(\sum_{n=1}^{\infty} \int_{x\in\Omega} \left(\frac{\partial}{\partial y} f_n(x,y) \right) dx \right) dy = \int_{x\in\Omega} \left(\sum_{n=1}^{\infty} f_n(x,\eta) \right) dx - \int_{x\in\Omega} \left(\sum_{n=1}^{\infty} f_n(x,c) \right) dx.$$

Differentiating both sides of (‡) with respect to η and applying the fundamental theorem of calculus in the theory of Lebesgue and then replacing η by y, we obtain

$$\sum_{n=1}^{\infty} \int_{x\in\Omega} \left(\frac{\partial}{\partial y} f_n(x,y) \right) dx = \frac{\partial}{\partial y} \int_{x\in\Omega} \left(\sum_{n=1}^{\infty} f_n(x,y) \right) dx$$

for almost all $y \in (a,b)$.

Solutions of Qualifying Exams III, 2014 Spring

1. (Algebra) Prove or disprove: There exists a prime number p such that the principal ideal (p) in the ring of integers $\mathcal{O}_K$ in $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a prime ideal.

Solution. If there were, the decomposition group and the inertia group at (p) would be isomorphic to the whole $\mathrm{Gal}\,(K/\mathbb{Q}) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ and to the trivial group, respectively, and the quotient would not be cyclic.

2. (Algebraic Geometry) Let $\Gamma = \{p_1, \cdots, p_5\} \subset \mathbb{P}^2$ be a configuration of 5 points in the plane.

(a) What is the smallest Hilbert function Γ can have?

(b) What is the largest Hilbert function Γ can have?

(c) Find all the Hilbert functions Γ can have.

Solution. (a) The smallest Hilbert function Γ can have occurs if Γ consists of 5 collinear points; the Hilbert function in this case is

$$(h_\Gamma(0), h_\Gamma(1), h_\Gamma(2), \dots) = (1, 2, 3, 4, 5, 5, \dots).$$

(b) The largest Hilbert function Γ can have occurs if Γ consists of 5 general points; the Hilbert function in this case is $(1, 3, 5, 5, \dots)$.

(c) The only other Hilbert function Γ can have occurs when Γ consists of four collinear points and one point not collinear with those; the Hilbert function in this case is $(1, 3, 4, 5, 5, \dots)$.

3. (Complex Analysis) *(Cauchy's Integral Formula for Smooth Functions and Solution of $\bar{\partial}$ Equation).* (a) Let Ω be a bounded domain in $\mathbb{C}$ with smooth boundary $\partial\Omega$. Let f be a C^∞ complex-valued function on some open neighborhood U of the topological closure $\bar{\Omega}$ of Ω in $\mathbb{C}$.

(i) Show that for $a \in \Omega$,

$$f(a) = \frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{f(z)dz}{z-a} + \frac{1}{2\pi i} \int_\Omega \frac{\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}}{z-a},$$

where

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + \sqrt{-1}\, \frac{\partial f}{\partial y} \right)$$

with $z = x + \sqrt{-1}\, y$ and x, y real.

(ii) Show that $a \in \Omega$,

$$f(a) = -\frac{1}{2\pi i}\int_{z\in\partial\Omega}\frac{f(z)d\bar{z}}{\bar{z}-\bar{a}} + \frac{1}{2\pi i}\int_{\Omega}\frac{\frac{\partial f}{\partial z}dz\wedge d\bar{z}}{\bar{z}-\bar{a}},$$

where

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \sqrt{-1}\,\frac{\partial f}{\partial y}\right).$$

(iii) For $z \in \Omega$ define

$$h(z) = \frac{1}{2\pi i}\int_{\zeta\in\Omega}\frac{f(\zeta)\,d\zeta\wedge d\bar{\zeta}}{\zeta - z}.$$

Show that $\frac{\partial h}{\partial \bar{z}}(z) = f(z)$ on Ω.

Hint: For (i), apply Stokes's theorem to $d\left(f(z)\frac{dz}{z-a}\right)$ on Ω minus a closed disk of radius $\varepsilon > 0$ centered at a and then let $\varepsilon \to 0$.

For the proof of (iii), for any fixed $z \in \Omega$, apply Stokes's theorem to $d\left(f(\zeta)\log|\zeta - z|d\bar{\zeta}\right)$ (with variable ζ) on Ω minus a closed disk of radius $\varepsilon > 0$ centered at z and then let $\varepsilon \to 0$. Then apply $\frac{\partial}{\partial\bar{z}}$ and use (ii).

(b) Let $\mathbb{D}_r$ be the open disk of radius $r > 0$ in $\mathbb{C}$ centered at 0. Prove that for any C^∞ complex-valued function g on $\mathbb{D}_1$ there exists some C^∞ complex-valued function h on $\mathbb{D}_1$ such that $\frac{\partial h}{\partial \bar{z}} = g$ on $\mathbb{D}_1$.

Hint: First use (a)(iii) to show that for $0 < r < 1$ there exists some C^∞ complex-valued function h_r on $\mathbb{D}_1$ such that $\frac{\partial h_r}{\partial \bar{z}} = g$ on $\mathbb{D}_r$. Then use some approximation and limiting process to construct h.

Solution. (a) Take an arbitrary positive number ε less than the distance from a to the boundary of Ω. Let B_ε be the closed disk of radius $\varepsilon > 0$ centered at a. Application of Stokes's theorem to

$$d\left(f(z)\frac{dz}{z-a}\right) = \frac{\frac{\partial f}{\partial \bar{z}}d\bar{z}\wedge dz}{z-a}$$

on $\Omega - B_\varepsilon$ yields

$$\int_{\Omega - B_\varepsilon}\frac{\frac{\partial f}{\partial\bar{z}}d\bar{z}\wedge dz}{z-a} = \int_{\partial\Omega} f(z)\frac{dz}{z-a} - \int_{|z-a|=\varepsilon} f(z)\frac{dz}{z-a}.$$

We use the parametrization $z = a + \varepsilon e^{i\theta}$ to evaluate the last integral and use $f\left(a + \varepsilon e^{i\theta}\right) - f(a) \to 0$ as $\varepsilon \to 0$ from the continuous differentiability of f at 0 to conclude that

$$\lim_{\varepsilon \to 0^+} \int_{|z-a|=\varepsilon} f(z) \frac{dz}{z-a} = 2\pi i f(a).$$

Hence

$$f(a) = \frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{f(z)dz}{z-a} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}}{z-a}.$$

This finishes the proof of the formula in (i). For the proof of the formula in (ii) we apply (i) to $\overline{f(z)}$ to get

$$\overline{f(a)} = \frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{\overline{f(z)}dz}{z-a} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial \overline{f}}{\partial \bar{z}} dz \wedge d\bar{z}}{z-a}.$$

and then we take the complex-conjugates of both sides to get

$$f(a) = -\frac{1}{2\pi i} \int_{z \in \partial\Omega} \frac{f(z)d\bar{z}}{\bar{z}-\bar{a}} + \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial f}{\partial z} dz \wedge d\bar{z}}{\bar{z}-\bar{a}},$$

which is the formula in (ii).

For the proof of (iii) we apply Stokes's theorem to

$$d\left(f(\zeta) \log |\zeta - z|^2 d\bar{\zeta}\right) = \frac{\partial f}{\partial \zeta} \log |\zeta - z|^2 d\zeta \wedge d\bar{\zeta} + \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

on $\Omega - B_\varepsilon$ yields

$$\begin{aligned}&\int_{\Omega - B_\varepsilon} \frac{\partial f}{\partial \zeta} \log |\zeta - z|^2 d\zeta \wedge d\bar{\zeta} + \int_{\Omega - B_\varepsilon} \frac{f(\zeta) d\zeta \wedge d\bar{\zeta}}{\zeta - z} \\ &= \int_{\partial\Omega} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta} - \int_{|z-a|=\varepsilon} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta}.\end{aligned}$$

With its evaluation by the parametrization $z = a + \varepsilon e^{i\theta}$, the last integral

$$\int_{|z-a|=\varepsilon} f(\zeta) \log |\zeta - z|^2 d\bar{\zeta}$$

approaches 0 as $\varepsilon \to 0+$ so that

$$\int_\Omega \frac{\partial f}{\partial \zeta} \log|\zeta - z|^2 d\zeta \wedge d\bar\zeta + \int_\Omega \frac{f(\zeta) d\zeta \wedge d\bar\zeta}{\zeta - z} = \int_{\partial\Omega} f(\zeta) \log|\zeta - z|^2 d\bar\zeta.$$

We apply $\frac{\partial}{\partial \bar z}$ to both sides and separately justify the commutation of $\frac{\partial}{\partial x}$ with integration and the commutation of $\frac{\partial}{\partial y}$ with integration, because on the right-hand sides of the following two formulae both integrals over Ω after differentiation are absolutely convergent.

$$\frac{\partial}{\partial x} \int_\Omega \frac{\partial f}{\partial \zeta} \log|\zeta - z|^2 d\zeta \wedge d\bar\zeta = \int_\Omega \frac{\partial f}{\partial \zeta} \left(\frac{\partial}{\partial x} \log|\zeta - z|^2 \right) d\zeta \wedge d\bar\zeta$$

and

$$\frac{\partial}{\partial y} \int_\Omega \frac{\partial f}{\partial \zeta} \log|\zeta - z|^2 d\zeta \wedge d\bar\zeta = \int_\Omega \frac{\partial f}{\partial \zeta} \left(\frac{\partial}{\partial y} \log|\zeta - z|^2 \right) d\zeta \wedge d\bar\zeta.$$

We get

$$-\int_\Omega \frac{\frac{\partial f}{\partial \zeta} d\zeta \wedge d\bar\zeta}{\bar\zeta - \bar z} + \frac{\partial}{\partial z} \int_\Omega \frac{f(\zeta) d\zeta \wedge d\bar\zeta}{\zeta - z} = -\int_{\partial\Omega} \frac{f(\zeta)\, d\bar\zeta}{\bar\zeta - \bar z},$$

or

$$\frac{\partial}{\partial z} \left(\frac{1}{2\pi i} \int_\Omega \frac{f(\zeta) d\zeta \wedge d\bar\zeta}{\zeta - z} \right) = -\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)\, d\bar\zeta}{\bar\zeta - \bar z} + \frac{1}{2\pi i} \int_\Omega \frac{\frac{\partial f}{\partial \zeta} d\zeta \wedge d\bar\zeta}{\bar\zeta - \bar z},$$

which by the formula in (ii) is equal to $f(z)$. This finishes the proof of the formula in (iii).

For use in (b) we also observe that

$$\frac{\partial}{\partial \bar z} \left(\frac{1}{2\pi i} \int_\Omega \frac{f(\zeta) d\zeta \wedge d\bar\zeta}{\zeta - z} \right) = -\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)\, d\bar\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_\Omega \frac{\frac{\partial f}{\partial \zeta} d\zeta \wedge d\bar\zeta}{\zeta - z}.$$

This implies that

$$\frac{\partial}{\partial \bar z} \left(\frac{1}{2\pi i} \int_\Omega \frac{f(\zeta) d\zeta \wedge d\bar\zeta}{\zeta - z} \right)$$

is uniformly bounded on compact subsets of Ω. By induction on k and by applying the argument to $\frac{\partial f}{\partial \zeta}$ on a neighborhood of $\bar\Omega$ in U in going from the

k-th step to the $(k+1)$-st step in the induction process, we conclude that all the k-th partial derivatives of

$$\int_{\Omega} \frac{f(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

with respect to x and y (*i.e.*, with respect to z and $\bar{z}$) are uniformly bounded on compact subsets of Ω. Hence

$$\int_{\Omega} \frac{f(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

is C^∞ on Ω as a function of z.

(b) Choose $r_n = 1 - \frac{1}{2^n}$. We can set

$$h_{r_n}(z) = \frac{1}{2\pi i} \int_{\zeta \in \mathbb{D}_{r_{n+1}}} \frac{g(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z}$$

on $\mathbb{D}_{r_n}$ to get $\partial_{\bar{z}} h_{r_n} = g$ on $\mathbb{D}_{r_n}$ from (a)(iii). As observed above, $h_{r_n}(z)$ is an infinitely differentiable function on $\mathbb{D}_{r_n}$.

We now look at the approximation and limiting process to construct h on all of $\mathbb{D}_1$ such that $\partial_{\bar{z}} h = g$ on $\mathbb{D}_1$.

For $n \geq 3$ the function $h_{r_n} - h_{r_{n-1}}$ is holomorphic on $\mathbb{D}_{r_{n-1}}$. By using the Taylor polynomial P_n of $h_{r_n} - h_{r_{n-1}}$ centered at 0 of degree N_n for N_n sufficiently large, we have

$$\left| \left(h_{r_n} - h_{r_{n-1}} \right) - P_n \right| \leq \frac{1}{2^n}$$

on $\mathbb{D}_{r_{n-2}}$. Let $\hat{h}_{r_n} = h_{r_n} - \sum_{k=3}^{n} P_n$ on $\mathbb{D}_{r_n}$. Then for any $n > k \geq 3$ from

$$\hat{h}_{r_n} - \hat{h}_{r_k} = \sum_{\ell=k+1}^{n} \left(\hat{h}_{r_\ell} - \hat{h}_{r_{\ell-1}} \right) = \sum_{\ell=k+1}^{n} \left(h_{r_\ell} - h_{r_{\ell-1}} - P_\ell \right)$$

it follows that

$$\left| \hat{h}_{r_n} - \hat{h}_{r_k} \right| \leq \sum_{\ell=k+1}^{n} \left| h_{r_\ell} - h_{r_{\ell-1}} - P_\ell \right| \leq \sum_{\ell=k+1}^{n} \frac{1}{2\ell} \leq \frac{1}{2^k}$$

on $\mathbb{D}_{r_{k-1}}$. Thus, for any fixed $k \geq 3$ the sequence $\{h_{r_n} - h_{r_k}\}_{n=k+1}^{\infty}$ is a Cauchy sequence of holomorphic functions on $\mathbb{D}_{r_{k-1}}$ and we can define $h = \lim_{n\to\infty} \hat{h}_{r_n}$ on $\mathbb{D}$ with $\frac{\partial h}{\partial \bar{z}} = g$ on $\mathbb{D}$, because $h - h_{r_k}$ is holomorphic on $\mathbb{D}_{r_{k-1}}$ and $\frac{\partial h_{r_{k-1}}}{\partial \bar{z}} = g$ on $\mathbb{D}_{r_{k-1}}$. Since $\hat{h}_{r_n}$ is infinitely differentiable on $\mathbb{D}_{r_{k-1}}$, it follows that h is infinitely differentiable on each $\mathbb{D}_{r_{k-1}}$ and hence is infinitely differentiable on all of $\mathbb{D}_1$.

4. (Algebraic Topology) Suppose that X is contractible and that some point a of X has a neighborhood homeomorphic to $\mathbb{R}^k$. Prove that $H_n(X \setminus \{a\}) \simeq H_n(S^{k-1})$ for all n.

Solution. We have the following piece of the long exact homology sequence:

$$H_k(X) \to H_k(X, X \setminus \{a\}) \to H_{k-1}(X \setminus \{a\}) \to H_{k-1}(X).$$

Now for $k > 1$, the outer two groups are 0, hence

$$H_k(X, X \setminus \{a\}) \simeq H_{k-1}(X \setminus \{a\}).$$

Let U be a neighborhood of a homeomorphic to $\mathbb{R}^m$ and let $C = X \setminus U$. Then $C \subset X \setminus \{a\}$, which is open. Hence, by excision,

$$H_k(X, X \setminus \{a\}) \simeq H_k(U, U \setminus \{a\}) \simeq H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{a\}).$$

On the other hand, we have the same piece of exact sequence of $(\mathbb{R}^m, \mathbb{R}^m \setminus \{a\}) : \psi(x, y) = (x, y)$ when $y < 0$, and

$$H_k(\mathbb{R}^m) \to H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{a\}) \to H_{k-1}(\mathbb{R}^m \setminus \{a\}) \to H_{k-1}(\mathbb{R}^m),$$

and the outer two groups are 0 for $k > 1$. Since $\mathbb{R}^m \setminus \{a\}$ deformation retracts onto S^{m-1}, putting everything together we obtain that for $k > 1$, $H_k(S^{m-1}) \simeq H_k(X \setminus \{a\})$.

5. (Differential Geometry) Let $U_+ = \mathbb{R}^2 - (\mathbb{R}_{\leq 0} \times \{0\})$, $U_- = \mathbb{R}^2 - (\mathbb{R}_{\geq 0} \times \{0\})$, and $U_0 = \mathbb{R}^2 - (\mathbb{R} \times \{0\})$. Let B be obtained by gluing U_+ and U_- over U_0 by the map $\psi : U_0 \to U_0$ defined by

$$\psi(x, y) = (x, y)$$

when $y < 0$, and

$$\psi(x, y) = (x + y, y)$$

when $y > 0$.

1. Show that B is a manifold.

2. Show that the trivial connections on the tangent bundles of U_+ and U_- glue together and give a global connection on the tangent bundle TB. Compute the curvature of this connection.

3. Compute the holonomy of the above connection around the loop $\gamma : [0, 2\pi] \to B$ determined by $\gamma|_{U_+}(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in (0, 2\pi)$.

Sketched Solution.

1. U_+ and U_- already serve as charts of B, and the transition between them is affine.

2. Since the transition is affine, the differential d is preserved by the transition. The curvature is just zero.

3. The holonomy is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

6. (Real Analysis) *(Bernstein's Theorem on Approximation of Continuous Functions by Polynomials).* Use the probabilistic argument outlined in the two steps below to prove the following theorem of Bernstein. Let f be a real-valued continuous function on $[0, 1]$. For any positive integer n let

$$B_n(f; x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right)\binom{n}{j} x^j (1-x)^{n-j}$$

be the *Bernstein polynomial.* Then $B_n(f; x)$ converges to f uniformly on $[0, 1]$ as $n \to \infty$.

Step One. For $0 < x < 1$ consider the binomial distribution

$$b(n, x, j) = \binom{n}{j} x^j (1-x)^{n-j}$$

for $0 \le j \le n$, which is the probability of getting j heads and $n - j$ tails in tossing a coin n times if the probability of getting a head is x. Verify that the mean μ of this probability distribution is nx and its standard deviation σ is $\sqrt{nx(1-x)}$.

Step Two. Let X be the random variable which assumes the value j with probability $b(n,x,j)$ for $0 \le j \le n$. Consider the random variable $Y = \left|f(x) - f\left(\frac{X}{n}\right)\right|$ which assumes the value $\left|f(x) - f\left(\frac{j}{n}\right)\right|$ with probability $b(n,x,j)$ for $0 \le j \le n$. Prove Bernstein's theorem by bounding, for an arbitrary positive number ε, the sum which defines the expected value $E(Y)$ of the random variable Y, after breaking the sum up into two parts defined respectively by $|j-\mu| \ge \eta\sigma$ and $|j-\mu| < \eta\sigma$ for some appropriate positive number η depending on ε and the uniform bound of f.

Solution. *Step One.* From

$$j\binom{n}{j} = n\,\frac{(n-1)(n-2)\cdots(n-j+1)}{(j-1)!} = n\binom{n-1}{j-1}$$

it follows that

$$\begin{aligned}
\mu &= \sum_{j=0}^{n} j\,b(n,x,j) \\
&= \sum_{j=0}^{n} j\binom{n}{j} x^j(1-x)^{n-j} \\
&= \sum_{j=1}^{n} n\binom{n-1}{j-1} x\,x^{j-1}(1-x)^{n-j} \\
&= nx\,(x+(1-x))^{n-1} \\
&= nx.
\end{aligned}$$

From

$$j(j-1)\binom{n}{j} = n(j-1)\binom{n-1}{j-1} = n(n-1)\binom{n-2}{j-2}$$

and

$$
\begin{aligned}
E(X(X-1)) = \sum_{j=0}^{n} j\,(j-1)\,b(n,x,j) &= \sum_{j=0}^{n} j\,(j-1)\binom{n}{j} x^j(1-x)^{n-j} \\
&= \sum_{j=2}^{n} n(n-1)x^2 \binom{n-2}{j-2} x^{j-2}(1-x)^{n-j} \\
&= n(n-1)x^2 \sum_{j=0}^{n-2} \binom{n-2}{j} x^j(1-x)^{n-2-j} \\
&= n(n-1)x^2\,(1+(1-x))^{n-2} \\
&= n(n-1)x^2
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\sigma^2 &= E\left((X-\mu)^2\right) \\
&= E(X^2) - 2\mu E(X) + \mu^2 \\
&= E(X^2) - \mu^2 \\
&= n(n-1)x^2 + nx - (nx)^2 \\
&= nx\left((n-1)x + 1 - nx\right) \\
&= nx(1-x).
\end{aligned}
$$

and $\sigma = \sqrt{nx(1-x)}$.

Step Two. Given any $\varepsilon > 0$. By the uniform continuity of f on $[0,1]$ there exists some $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for $|x_1 - x_2| < \delta$. Choose a positive number η sufficiently large so that

$$\frac{1}{\eta^2}\, 2 \sup_{[0,1]} |f| < \frac{\varepsilon}{2}$$

and then choose a positive N with

$$\frac{\eta}{\sqrt{N}} < \delta.$$

We are going to prove that $|f - B_n(f;x)| < \varepsilon$ on $[0,1]$ for $n \geq N$ by bounding the sum which defines the expected value $E(Y)$ of the random variable Y, after breaking the sum up into two parts defined respectively by $|j-\mu| \geq \eta\sigma$ and $|j-\mu| < \eta\sigma$.

First of all, for any fixed $x \in [0,1]$,

$$\begin{aligned}|f(x) - B_n(f;x)| &= \left|\sum_{j=0}^{n} \left(f(x) - f\left(\frac{j}{n}\right)\right)\binom{n}{j} x^j (1-x)^{n-j}\right| \\ &= \left|\sum_{j=0}^{n} \left(f(x) - f\left(\frac{j}{n}\right)\right) b(n,x,j)\right| \\ &\leq \sum_{j=0}^{n} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n,x,j),\end{aligned}$$

which is the expected value $E(Y)$ of the random variable Y, because

$$\begin{aligned}\sum_{j=0}^{n} f(x)\binom{n}{j} x^j (1-x)^{n-j} &= f(x)\sum_{j=0}^{n}\binom{n}{j} x^j (1-x)^{n-j} \\ &= f(x)\,(x + (1-x))^n = f(x).\end{aligned}$$

For the estimation of the part

$$\sum_{|j-nx|<\eta\sigma} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n,x,j)$$

of the sum

$$E(Y) = \sum_{j=0}^{n} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n,x,j)$$

we have

$$\left|x - \frac{j}{n}\right| < \frac{\eta\sigma}{n} = \frac{\eta\sqrt{nx(1-x)}}{n} \leq \frac{\eta}{\sqrt{n}} \leq \frac{\eta}{\sqrt{N}} < \delta,$$

which implies that $\left|f(x) - f\left(\frac{j}{n}\right)\right| < \frac{\varepsilon}{2}$ so that

$$\sum_{|j-nx|<\eta\sigma} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n,x,j) < \frac{\varepsilon}{2}\, b(n,x,j) \leq \frac{\varepsilon}{2}.$$

For the estimation of the part

$$\sum_{|j-nx|\geq\eta\sigma} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n,x,j)$$

of the sum

$$E(Y) = \sum_{j=0}^{n} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j)$$

we use Chebyshev's inequality that in any probability distribution no more than $\frac{1}{\eta^2}$ of the distribution's values can be no less than η standard deviations away from the mean, which, when applied to our random variable X with mean $\mu = nx$, means that

$$\sum_{|j-nx|\geq \eta\sigma} b(n,x,j) \leq \frac{1}{\eta^2}.$$

Thus,

$$\sum_{|j-nx|\geq \eta\sigma} \left| f(x) - f\left(\frac{j}{n}\right) \right| b(n,x,j) \leq \left(2 \sup_{[0,1]} |f| \right) \frac{1}{\eta^2} < \frac{\varepsilon}{2}.$$

This finishes the verification that

$$|f(x) - B_n(f;x)| < \varepsilon$$

for $n \geq N$ and thus the proof of Bernstein's theorem.