QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday, January 18, 2011 (Day 1)

1. (CA) Evaluate

$$\int_0^\infty \frac{x^2+1}{x^4+1}\mathrm{d}x$$

2. (A) Let k be a field and V be a k-vector space of dimension n. Let $A \in End_k(V)$. Show that the following are equivalent:

   (a) The minimal polynomial of A is the same as the characteristic polynomial of A.

   (b) There exists a vector $v \in V$ such that $v, Av, A^2v, \ldots, A^{n-1}v$ is a basis of V.

3. (T) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

4. (RA)

   (a) Prove that any countable subset of the interval $[0,1] \subset \mathbb{R}$ is Lebesgue measurable, and has Lebesgue measure 0.

   (b) Let $\Phi \subset [0,1]$ be the set of real numbers x that, when written as a decimal $x = 0.a_1a_2a_3\ldots$, satisfy the rule $a_{n+2} \notin \{a_n, a_{n+1}\}$ for all $n \geq 1$. What is the Lebesgue measure of Φ?

5. (DG) Let $B \subset \mathbb{R}^4$ be the closed ball of radius 2 centered at the origin, with the metric induced from the euclidean metric on $\mathbb{R}^4$. Give an example of a smooth vector field v on B with the property that for any L there exists an integral curve of v with both endpoints on the boundary ∂B and length greater than L.

6. (AG) Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, non-degenerate curve of degree d.

   (a) Show that $d \geq 3$.

   (b) Show that every point $p \in \mathbb{P}^3$ lies on a secant or tangent line to C.

   (c) If $d = 3$, show that every point of $\mathbb{P}^3 \backslash C$ lies on a unique secant or tangent line to C.

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Wednesday, January 19, 2011 (Day 2)

1. (T) Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$ up to isomorphism of covering spaces without basepoints. Indicate which covering spaces are normal.

2. (RA) Let g be a differentiable function on $\mathbb{R}$ that is non-negative and has compact support.

   (a) Prove that the Fourier transform $\hat{g}$ of g does not have compact support unless $g = 0$.

   (b) Prove that there exist constants A and c such that for all $k \in \mathbb{N}$ the k^{th} derivative of $\hat{g}$ is bounded by cA^k.

3. (DG) Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius 1 centered at the origin, with the metric induced from the euclidean metric on $\mathbb{R}^3$. Introduce spherical coordinates $(\theta, \phi) \in [0, \pi] \times \mathbb{R}/(2\pi\mathbb{Z})$ on the complement of the north and south poles, where

$$(x, y, z) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta).$$

   The metric in these coordinates is given by the section

$$d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi$$

   of the second symmetric power of the cotangent bundle T^*S^2; it has constant scalar curvature 1.

   Now let u be a smooth function on S^2 depending only on the coordinate θ, and consider the metric given by the section

$$e^u\big(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi\big).$$

   (a) Compute the scalar curvature of this new metric in terms of u and its derivatives.

   (b) Prove that the integral over S^2 of the function you computed in Part (a) is equal to 4π.

4. (AG) Show that no two of the following rings are isomorphic:

   1. $\mathbb{C}[x,y]/(y^2 - x)$
   2. $\mathbb{C}[x,y]/(y^2 - x^2)$
   3. $\mathbb{C}[x,y]/(y^2 - x^3)$
   4. $\mathbb{C}[x,y]/(y^2 - x^4)$
   5. $\mathbb{C}[x,y]/(y^2 - x^5)$
   6. $\mathbb{C}[x,y]/(y^3 - x^4)$

5. (CA) Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

6. (A) Let a be a positive integer, and consider the polynomial

$$f_a(x) = x^6 + 3ax^4 + 3x^3 + 3ax^2 + 1 \in \mathbb{Q}[x].$$

   (a) Show that it is irreducible.

   (b) Show that the Galois group of f_a is solvable.

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Thursday, January 20, 2011 (Day 3)

1. (DG) Let $(\,\cdot\,)$ be the standard inner product on $\mathbb{R}^3$, and let

$$S^2 = \{\mathbf{x} = (x_1, x_2, x_3) : (\mathbf{x} \cdot \mathbf{x}) = 1\}$$

be the sphere of radius 1 centered at the origin; identify the tangent space $T_{\mathbf{x}}S^2$ at a point $\mathbf{x} \in S^2$ with the subspace

$$T_{\mathbf{x}}S^2 = \{v \in \mathbb{R}^3 : (\mathbf{x} \cdot v) = 0\} \ \subset \ \mathbb{R}^3,$$

where $(\,\cdot\,)$ is the standard inner product on $\mathbb{R}^3$. Let $e \in \mathbb{R}^3$ be any fixed vector, and let V be the vector field on S^2 given by

$$V(\mathbf{x}) = e - (\mathbf{x} \cdot e)\mathbf{x}.$$

(a) Compute the Lie derivative by V of the 1-form $x_1 dx_2$.

(b) Define a Riemannian metric on S^2 by setting the inner product of tangent vectors $v, v' \in T_{\mathbf{x}}S^2$ equal to $(v \cdot v')$ (that is, take the metric induced on S^2 by the euclidean metric on $\mathbb{R}^3$). Use the associated Levi-Civita connection to define a covariant derivative on the space of 1-forms on S^2.

(c) Compute the covariant derivative of the 1-form $x_1 dx_2$ in the direction of the vector field V.

2. (T) Let D^2 be the closed unit disk in $\mathbb{R}^2$. Prove the Brouwer fixed point theorem for maps $f : D^2 \to D^2$ by applying degree theory to the map $S^2 \to S^2$ that sends both the northern and southern hemispheres of S^2 to the southern hemisphere via f.

3. (CA) Prove that for every $\lambda > 1$, the equation $ze^{\lambda - z} = 1$ has exactly one root in the unit disk $\mathbb{D}$ and that this root is real.

4. (A) Let K be an algebraically closed field of characteristic 0, and let $f \in K[x]$ be any cubic polynomial. Show that exactly one of the following two statements is true:

1. $f = \alpha(x-\lambda)^3 + \beta(x-\lambda)^2$ for some $\alpha, \beta, \lambda \in K$; or
2. $f = \alpha(x-\lambda)^3 + \beta(x-\mu)^3$ for some $\alpha, \beta \neq 0 \in K$ and $\lambda \neq \mu \in K$.

In the second case, show that λ and μ are unique up to order.

5. (AG) Let $Q \subset \mathbb{P}^{2n+1}$ be a smooth quadric hypersurface in an odd-dimensional projective space over $\mathbb{C}$.

   (a) What is the largest dimension of a linear subspace of $\mathbb{P}^{2n+1}$ contained in Q?

   (b) What is the dimension of the family of such planes?

6. (RA) Let $\mathbb{H}$ and $\mathbb{L}$ denote a pair of Banach spaces.

   (a) Prove that a linear map from $\mathbb{H}$ to $\mathbb{L}$ is continuous if and only if it's bounded.

   (b) Define what is meant by a *compact* linear map from $\mathbb{H}$ to $\mathbb{L}$.

   (c) Now let $\mathbb{H}$ and $\mathbb{L}$ be the Banach spaces obtained by completing the space $C_c^\infty([0,1])$ of compactly supported C^∞ functions on $[0,1]$ using the norms with squares

$$\|f\|_{\mathbb{H}}^2 = \int_{[0,1]} |\frac{df}{ds}|^2 s^2 ds \quad \text{and} \quad \|f\|_{\mathbb{L}}^2 = \int_{[0,1]} |f|^2 ds.$$

   The identity map $C_c^\infty([0,1])$ extends to a bounded linear map $\phi : \mathbb{H} \to \mathbb{L}$ (you don't need to prove this). Prove that ϕ is not compact.