QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 19, 2010 (Day 1)

1. Let (X, μ) be a measure space with $\mu(X) < \infty$. For $q > 0$, let $L^q = L^q(X, \mu)$ denote the Banach space completion of the space of bounded functions on X with the norm

$$||f||_q = \left(\int_X |f|^q \mu \right)^{\frac{1}{q}}.$$

Now suppose that $0 < p \leq q$. Prove that all functions in L^q are in L^p, and that the inclusion map $L^q \hookrightarrow L^p$ is continuous.

2. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k, $\mathbb{G}(\ell, n)$ the Grassmannian of ℓ-planes in $\mathbb{P}^n$ for some $\ell < n - k$, and $C(X) \subset \mathbb{G}(\ell, n)$ the variety of ℓ-planes meeting X. Prove that $C(X)$ is irreducible, and find its dimension.

3. Let λ be real number greater than 1. Show that the equation $ze^{\lambda - z} = 1$ has exactly one solution z with $|z| < 1$, and that this solution z is real. (Hint: use Rouché's theorem.)

4. Let k be a finite field, with algebraic closure $\overline{k}$.

(a) For each integer $n \geq 1$, show that there is a unique subfield $k_n \subset \overline{k}$ containing k and having degree n over k.

(b) Show that k_n is a Galois extension of k, with cyclic Galois group.

(c) Show that the norm map $k_n^\times \to k^\times$ (sending a nonzero element of k_n to the product of its Galois conjugates) is a surjective homomorphism.

5. Suppose ω is a closed 2-form on a manifold M. For every point $p \in M$, let

$$R_p(\omega) = \{v \in T_pM \ : \ \omega_p(v, u) = 0 \text{ for all } u \in T_pM\}.$$

Suppose that the dimension of R_p is the same for all p. Show that the assignment $p \mapsto R_p$ as p varies in M defines an integrable subbundle of the tangent bundle TM.

6. Let X be a topological space. We say that two covering spaces $f : Y \to X$ and $g : Z \to X$ are isomorphic if there exists a homeomorphism $h : Y \to Z$ such that $g \circ h = f$. If X is a compact oriented surface of genus g (that is, a g-holed torus), how many connected 2-sheeted covering spaces does X have, up to isomorphism?

QUALIFYING EXAMINATION

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Wednesday January 20, 2010 (Day 2)

1. Let a be an arbitrary real number and b a positive real number. Evaluate the integral

$$\int_0^\infty \frac{\cos(ax)}{\cosh(bx)} dx$$

(Recall that $\cosh(x) = \cos(ix) = \frac{1}{2}(e^x + e^{-x})$ is the hyperbolic cosine.)

2. For any irreducible plane curve $C \subset \mathbb{P}^2$ of degree $d > 1$, we define the *Gauss map* $g : C \to \mathbb{P}^{2^*}$ to be the rational map sending a smooth point $p \in C$ to its tangent line; we define the *dual curve* $C^* \subset \mathbb{P}^{2^*}$ of C to be the image of g.

(a) Show that the dual of the dual of C is C itself.

(b) Show that two irreducible conic curves $C, C' \subset \mathbb{P}^2$ are tangent if and only if their duals are.

3. Let Λ_1 and $\Lambda_2 \subset \mathbb{R}^4$ be complementary 2-planes, and let $X = \mathbb{R}^4 \setminus (\Lambda_1 \cup \Lambda_2)$ be the complement of their union. Find the homology and cohomology groups of X with integer coefficients.

4. Let $X = \{(x, y, z) : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ be a cylinder. Show that the geodesics on X are *helices*, that is, curves such that the angle between their tangent lines and the vertical is constant.

5. (a) Show that if every closed and bounded subspace of a Hilbert space E is compact, then E is finite dimensional.

(b) Show that any decreasing sequence of nonempty, closed, convex, and bounded subsets of a Hilbert space has a nonempty intersection.

(c) Show that any closed, convex, and bounded subset of a Hilbert space is the intersection of the closed balls that contain it.

(d) Deduce that any closed, convex, and bounded subset of a Hilbert space is compact in the weak topology.

6. Let p be a prime, and let G be the group $\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$.

(a) How many subgroups of order p does G have?

(b) How many subgroups of order p^2 does G have? How many of these are cyclic?

QUALIFYING EXAMINATION

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Thursday January 21, 2010 (Day 3)

1. Consider the ring

$$A = \mathbb{Z}[x]/(f) \quad \text{where} \quad f = x^4 - x^3 + x^2 - 2x + 4.$$

Find all prime ideals of A that contain the ideal (3).

2. Let f be a holomorphic function on a domain containing the closed disc $\{z : |z| \leq 3\}$, and suppose that

$$f(1) = f(i) = f(-1) = f(-i) = 0.$$

Show that

$$|f(0)| \leq \frac{1}{80} \max_{|z|=3} |f(z)|$$

and find all such functions for which equality holds in this inequality.

3. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a differentiable, positive real function. Find the Gaussian curvature and mean curvature of the surface of revolution

$$S = \{(x, y, z) : y^2 + z^2 = f(x)\}.$$

4. Show that for any given natural number n, there exists a finite Borel measure on the interval $[0, 1] \subset \mathbb{R}$ such that for any real polynomial of degree at most n, we have

$$\int_0^1 p \, d\mu = p'(0).$$

Show, on the other hand, that there does *not* exist a finite Borel measure on the interval $[0, 1] \subset \mathbb{R}$ such that for any real polynomial we have

$$\int_0^1 p \, d\mu = p'(0).$$

5. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^4$.

(a) Find the homology groups $H_*(X, \mathbb{Z}/2)$

(b) Find the homology groups $H_*(X, \mathbb{Z})$

(c) Find the cohomology groups $H^*(X, \mathbb{Z})$

6. By a *twisted cubic curve* we mean the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$[X, Y] \mapsto \big[F_0(X, Y), F_1(X, Y), F_2(X, Y), F_3(X, Y)\big]$$

where $F_0, \ldots, F_3$ form a basis for the space of homogeneous cubic polynomials in X and Y.

(a) Show that if $C \subset \mathbb{P}^3$ is a twisted cubic curve, then there is a 3-dimensional vector space of homogeneous quadratic polynomials on $\mathbb{P}^3$ vanishing on C.

(b) Show that C is the common zero locus of the homogeneous quadratic polynomials vanishing on it.

(c) Suppose now that $Q, Q' \subset \mathbb{P}^3$ are two distinct quadric surfaces containing C. Describe the intersection $Q \cap Q'$.