QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday January 29 2008 (Day 1)

1. Let $K = \mathbb{C}(x)$ be the field of rational functions in one variable over $\mathbb{C}$, and consider the polynomial

$$f(y) \;=\; y^4 + x \cdot y^2 + x \;\in\; K[y].$$

(a) Show that f is irreducible in $K[y]$.

(b) Let $L = K[y]/(f)$. Is L a Galois extension of K?

(c) Let L' be the splitting field of f over K. Find the Galois group of L'/K

2. Let f be a holomorphic function on the unit disc $\Delta = \{z : |z| < 1\}$. Suppose $|f(z)| < 1$ for all $z \in \Delta$, and that

$$f(\frac{1}{2}) = f(-\frac{1}{2}) = 0.$$

Show that $|f(0)| \leq \frac{1}{3}$.

3. Let $\mathbb{CP}^n$ be complex projective n-space.

(a) Describe the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$.

(b) Let $i : \mathbb{CP}^n \to \mathbb{CP}^{n+1}$ be the inclusion of $\mathbb{CP}^n$ as a hyperplane in $\mathbb{CP}^{n+1}$. Show that there does not exist a map $f : \mathbb{CP}^{n+1} \to \mathbb{CP}^n$ such that the composition $f \circ i$ is the identity on $\mathbb{CP}^n$.

4. Let f be the function on $\mathbb{R}$ defined by

$$f(t) = t, \quad -\pi < t \leq \pi$$

and

$$f(t + 2\pi) = f(t) \quad \forall t.$$

Find the Fourier expansion of f.

5. Let X, Y, Z and W be homogeneous coordinates on projective space $\mathbb{P}^3$ over a field K, and $Q \subset \mathbb{P}^3$ be the surface defined by the equation $XY - ZW = 0$.

(a) Show that Q is smooth and irreducible.

(b) Show that Q is birational to $\mathbb{P}^2$, that is, the function field of Q is isomorphic to $K(s, t)$.

(c) Show that Q is *not* isomorphic to $\mathbb{P}^2$.

6. (a) Define the *curvature* and *torsion* of a differentiable arc in $\mathbb{R}^3$.

(b) Let $\Delta \subset \mathbb{R}^3$ be an arc given parametrically by the $\mathcal{C}^\infty$ vector-valued function $t \mapsto v(t) \in \mathbb{R}^3$ for t in the interval $I = (-1, 1) \subset \mathbb{R}$. Under what conditions is the map

$$\phi : (-\epsilon, \epsilon) \times (0, \eta) \;\rightarrow\; \mathbb{R}^3$$

given by

$$\phi(t, s) \;\mapsto\; v(t) + s \cdot v'(t)$$

an immersion for some positive ϵ and η?

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Wednesday January 30 2008 (Day 2)

1. Let $X \subset \mathbb{R}^3$ be the cone $x^2 = y^2+z^2$, and let Y be the torus $(\sqrt{x^2+y^2}-2)^2 + z^2 = 1$, that is, the torus obtained by rotating the circle $(x-2)^2+z^2-1 = y = 0$ around the z-axis.

(a) Show that for any point $p \in X$ other than the vertex $(0,0,0)$, there is a neighborhood of p in X isometric to an open subset of the Euclidean plane $\mathbb{R}^2$.

(b) Show that no open subset of Y is isometric to any open subset of the Euclidean plane.

2. Let V be an n-dimensional vector space over a field K, and $Q : V \times V \to K$ a symmetric bilinear form. By the *kernel* of Q we mean the subspace of V of vectors v such that $Q(v,w) = 0$ for all $w \in V$, and by the *rank* of Q we mean n minus the dimension of the kernel of Q.

Let $W \subset V$ be a subspace of dimension $n-k$, and let Q' be the restriction of Q to W. Show that

$$\operatorname{rank}(Q) - 2k \;\le\; \operatorname{rank}(Q') \;\le\; \operatorname{rank}(Q).$$

3. Find the solution of the differential equation

$$y''' - y'' - y' + y \;=\; 0$$

satisfying the conditions

$$y(0) = y'(0) = 0 \quad \text{and} \quad y''(0) = 1.$$

4. Let S be a compact orientable 2-manifold of genus g, and let S_2 be its *symmetric square*, that is, the quotient of the ordinary product $S \times S$ by the involution exchanging factors.

(a) Show that S_2 is a manifold.

(b) Find the Euler characteristic $\chi(S_2)$.

(c) Find the Betti numbers of S_2.

5. Prove the identity

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$

6. Let $\mathbb{P} \cong \mathbb{P}^n$ be the space of nonzero homogeneous polynomials of degree n in two variables, mod scalars, and let $\Delta \subset \mathbb{P}$ be the locus of polynomials with a repeated factor.

(a) Show that Δ is an irreducible subvariety of $\mathbb{P}$.

(b) Show that $\dim \Delta = n - 1$.

(c) What is the degree of Δ?

QUALIFYING EXAMINATION

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Thursday January 31 2008 (Day 3)

1. For c a nonzero real number, evaluate the integral

$$\int_0^\infty \frac{\log z}{z^2 + c^2}\, dz$$

2. Let $\mathbb{G}(1,4)$ be the Grassmannian parametrizing lines in $\mathbb{P}^4$, and let $Q \subset \mathbb{P}^4$ be a smooth quadric hypersurface. Let $F \subset \mathbb{G}(1,4)$ be the set of lines contained in Q.

(a) Show that F is an algebraic subvariety of $\mathbb{G}(1,4)$.

(b) Show that F is irreducible.

(c) What is the dimension of F?

3. Let S be a compact orientable 2-manifold of genus 2 (that is, a 2-holed torus), and let $f : S \to S$ be any orientation-preserving homeomorphism of finite order.

(a) Show that f must have a fixed point.

(b) Is this statement still true if we drop the hypothesis that f is orientation-preserving? Prove or give a counterexample.

(c) Is this statement still true if we replace S by a compact orientable 2-manifold of genus 3? Again, prove or give a counterexample

4. (a) State Fermat's Little Theorem on powers in the field $\mathbb{F}_{37}$ with 37 elements.

(b) Let k be any natural number not divisible by 2 or 3, and let $a \in \mathbb{F}_{37}$ be any element. Show that there exists a unique solution to the equation

$$x^k = a$$

in $\mathbb{F}_{37}$.

(c) Solve the equation

$$x^5 = 2$$

in $\mathbb{F}_{37}$.

5. Let X be a Banach space.

(a) Define the *weak topology* on X by describing a basis for the topology.

(b) Let $A : X \to Y$ be a linear operator between Banach spaces that is continuous from the weak topology on X to the norm topology on Y. Show that the image $A(X) \subset Y$ is finite-dimensional.

6. Let $V \cong \mathbb{C}^2$ be the standard representation of $SL_2(\mathbb{C})$.

(a) Show that the n^{th} symmetric power $V_n = \text{Sym}^n V$ is irreducible.

(b) Which V_n appear in the decomposition of the tensor product $V_2 \otimes V_3$ into irreducible representations?