QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday 10 February 2004 (Day 1)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1a. Prove the following theorem of Banach and Saks:

Theorem. *Given in L^2 a sequence $\{f_n\}$ which weakly converges to 0, we can select a subsequence $\{f_{n_k}\}$ such that the sequence of arithmetic means*

$$\frac{f_{n_1} + f_{n_2} + \cdots + f_{n_k}}{k}$$

strongly converges to 0.

(Recall: We say that the sequence $\{f_n\}$ *strongly converges* to f when $||f - f_n|| \to 0$. We say that the sequence $\{f_n\}$ *weakly converges* to f if for every $g \in L^2$, the sequence (f_n, g) converges to (f, g).)

2a. Fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. Let $\overline{\mathbf{Z}}$ denote the subset of all elements of $\overline{\mathbf{Q}}$ that satisfy a monic polynomial with coefficients in the ring $\mathbf{Z}$ of integers. You may assume that $\overline{\mathbf{Z}}$ is a ring.

(i) Show that the ideals (2) and $(\sqrt{2})$ in $\overline{\mathbf{Z}}$ are distinct.

(ii) Prove that $\overline{\mathbf{Z}}$ is not Noetherian.

3a. Let B denote the open unit disk in the complex plane $\mathbf{C}$.

(i) Does there exist a surjective, complex-analytic map $f : \mathbf{C} \to B$?

(ii) Does there exist a surjective, complex-analytic map $f : B \to \mathbf{C}$?

4a. (i) Draw a picture of a compact, orientable 2-manifold S (without boundary) of genus 2. On your picture, draw a base-point x and a simple closed curve γ on S that represents a non-trivial element of $\pi_1(S, x)$ but represents the zero element of $H_1(S)$. Justify your answer.

(ii) Let $p : \tilde{S} \to S$ be a two-to-one covering space. Let $\tilde{x}$ be one of the two points in $p^{-1}(x)$. Show that there is a closed path $\tilde{\gamma}$ based at $\tilde{x}$ in $\tilde{S}$ such that $\gamma = p \circ \tilde{\gamma}$.

5a. Given $0 < b < a$, define

$$g(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u),$$

for $(u,v) \in \mathbf{R} \times \mathbf{R}$. The image is a torus. Compute the Gaussian curvature of this torus at points $g(0,v)$.

6a. (i) Let X by a smooth hypersurface of degree d in $\mathbf{P}^n$. What is the degree of the projection of X from one of its points onto a general hyperplane in $\mathbf{P}^n$?

(ii) Prove that every smooth quadric hypersurface in $\mathbf{P}^n$ is rational. (A variety X is rational if it admits a birational map to some projective space.)

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Wednesday 11 February 2004 (Day 2)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1b. Let H be a Hilbert space, and let P be a subset of H (not necessarily a subspace). By the orthogonal complement of P we mean the set

$$P^\perp = \{y : y \perp x \text{ for all } x \in P\}.$$

(i) (4 points) Show that $P^\perp$ is always a closed vector subspace of H.

(ii) (6 points) Show that $P^{\perp\perp}$ is the smallest closed vector subspace that contains P.

2b. Prove that each of the following rings contains infinitely many prime ideals:

(i) (2 points) The ring $\mathbf{Z}$ of integers.

(ii) (2 points) The ring $\mathbf{Q}[x]$ of polynomials over $\mathbf{Q}$.

(iii) (3 points) The ring of regular functions on an affine algebraic surface over $\mathbf{C}$. (You may assume standard results from algebraic geometry.)

(iv) (3 points) The countable direct product of copies of R, for any nonzero commutative ring R with unity.

3b. Show that if $f(z)$ is a polynomial of degree at least 2, then the sum of the residues of $1/f(z)$ at all the zeros of $f(z)$ must be 0.

4b. Let X be a smooth, compact, oriented manifold.

(i) (4 points) Give a clear statement of the Poincaré duality theorem as it applies to the singular homology of X. Deduce from the duality theorem that, if X is connected, there is an isomorphism $\epsilon : H^n(X) \to \mathbf{Z}$.

(ii) (6 points) Use the universal coefficient theorem and the Poincaré duality theorem to show that, if $a \in H^i(X)$ is not a torsion element, then there exists $b \in H^{n-i}(X)$ such that the cup product $a \smile b$ is nonzero.

5b. Let $M^2 \subset \mathbf{R}^3$ be an embedded oriented surface and let S^2 be the unit sphere. The Gauss map $G : M \to S^2$ is defined to be $G(x) = \vec{N}(x)$ for any $x \in M$, where $\vec{N}(x)$ is the unit normal vector of M at x. Let h and g denote the induced Riemannian metric on M and S^2 from $\mathbf{R}^3$ respectively. Prove that if the mean curvature of M is zero everywhere, then the Gauss map G is a conformal map from (M, h) to (S^2, g).

(Recall: If (Σ_1, g_1), (Σ_2, g_2) are two Riemannian manifolds, a map $\varphi : (\Sigma_1, g_1) \to (\Sigma_2, g_2)$ is called *conformal* if $g_1 = \lambda \varphi^* g_2$ for some scalar function λ on Σ_1.)

6b. Consider $X = \mathbf{P}^1 \times \mathbf{P}^1$ sitting in $\mathbf{P}^3$ via the Segre embedding. Prove that the Zariski topology on X is different from the product topology induced by the Zariski topology on both $\mathbf{P}^1$ factors.

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Thursday 12 February 2004 (Day 3)

There are six problems. Each question is worth 10 points, and parts of questions are of equal weight unless otherwise specified.

1c. Let f be a continuous, real-valued, increasing function on an interval $[a, b]$ such that $f(a) > a$ and $f(b) < b$. Let $x_1 \in [a, b]$, and define a sequence via $x_n = f(x_{n-1})$. Show that $\lim_{n\to\infty} x_n$ exists. If we call this number x^*, show that $f(x^*) = x^*$.

2c. Describe all the irreducible complex representations of the group S_4 (the symmetric group on four letters).

3c. Suppose f is a biholomorphism between two closed annuli in $\mathbf{C}$

$$A(R) = \{z \in \mathbf{C} \mid 1 \leq |z| \leq R\} \qquad \text{and} \qquad A(S) = \{z \in \mathbf{C} \mid 1 \leq |z| \leq S\},$$

with $R, S > 1$.

(i) Show that f can be extended to a biholomorphic map from $\mathbf{C} \setminus \{0\}$ to $\mathbf{C} \setminus \{0\}$.

(ii) Prove that $R = S$.

4c. Use homotopy groups to show that there is no retraction $r : \mathbf{RP}^n \to \mathbf{RP}^k$ if $n > k > 0$. (Here $\mathbf{RP}^n$ is real projective space of dimension n.)

5c. Let $\alpha : I \to \mathbf{R}^3$ be a regular curve with nonzero curvature everywhere. Show that if the torsion $\tau(t) = 0$ for all $t \in I$, then $\alpha(t)$ is a plane curve (i.e., the image of α lies entirely in a plane).

6c. Let X be a k-dimensional irreducible subvariety of $\mathbf{P}^n$. In the Grassmannian $\mathbf{G}(1, n)$ of lines in $\mathbf{P}^n$, let $S(X)$ be the set of lines which are secant to X, i.e., which meet X in at least two distinct points. Consider also the union $C(X)$ of all these secant lines, which is a subset of $\mathbf{P}^n$.

(i) Prove that if X is not a linear subspace of $\mathbf{P}^n$, then the closure of $S(X)$ is an irreducible subvariety of $\mathbf{G}(1, n)$ of dimension $2k$.

(ii) Prove that the closure of $C(X)$ is an irreducible subvariety of $\mathbf{P}^n$ of dimension at most $2k + 1$.