QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Tuesday September 2, 2014 (Day 1)

1. (AG) For any $0 < k < m \leq n \in \mathbb{Z}$, let $M \cong \mathbb{P}^{mn-1}$ be the space of nonzero $m \times n$ matrices mod scalars, and let $M_k \subset M$ be the subset of matrices of rank k or less.

(a) Show that M_k is closed in M (in the Zariski topology).

(b) Show that M_k is irreducible.

(c) What is the dimension of M_k?

(d) What is the degree of M_1?

Solution: For the first, M_k is the zero locus of the $(k+1) \times (k+1)$ minors, which are homogeneous polynomials of degree $k+1$ on $M \cong \mathbb{P}^{mn-1}$. For the second and third, we introduce the incidence correspondence

$$\Phi = \{(\Lambda, A) \in G(n-k, n) \mid \Lambda \subset \ker(A)\}.$$

Since Φ is fibered over $G(n-k, n)$ with fibers $\mathbb{P}^{km-1}$, it is irreducible of dimension $k(n-k)+km-1 = mn-1-(m-k)(n-k)$; since it is generically one-to-one over M_k, we conclude that M_k is likewise irreducible of that dimension. Finally, M_1 is the Segre variety $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \subset \mathbb{P}^{mm-1}$, which has degree $\binom{m+n-2}{m-1}$.

2. (A) Let S_3 be the group of automorphisms of a 3-element set.

(a) Classify the conjugacy classes of S_3.

(b) Classify the irreducible representations of S_3.

(c) Write the character table for S_3.

Solution: (a) Conjugacy classes of symmetric groups are given by the types of cycles one can write on the set of n elements. For $n = 3$, we have shapes given by $(1), (12), (123)$ so we have three conjugacy classes. (b) By the orthogonality relations, the number of irreps are equal to the number of conjugacy classes. On the other hand, we can produce three irreps: The trivial, the sign, and the geometric representation corresponding to $S_3 \cong D_6$; i.e., the symmetries of an equilateral triangle embedded in $\mathbb{R}^2$. (c) Compute the traces of each conjugacy class. End up with the table

	(1)	(12)	(123)
triv	1	1	1
sign	1	-1	1
geom	2	0	-1

3. (DG) Let x, y, z be the standard coordinates on $\mathbb{R}^3$. Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

1. Compute the critical points of the function $x|_{\mathbb{S}^2}$. Show that they are isolated and non-degenerate.
2. Equip $\mathbb{S}^2$ with the standard metric induced from $\mathbb{R}^3$. Compute the gradient vector field of $x|_{\mathbb{S}^2}$. Compute the integral curves of this vector field.

Solution:

1. The unit sphere is defined by $x^2 + y^2 + z^2 = 1$. Regarding y, z as independent variables and x as dependent variable, we have the equations $2x\partial_y x + 2y = 0$ and $2x\partial_z x + 2z = 0$. For a critical point, $\partial_y x = \partial_z x = 0$, and hence $y = z = 0$. Then $x = \pm 1$. Hence the critical points are $(1, 0, 0)$ and $(-1, 0, 0)$.
   They are isolated in $\mathbb{S}^2$. Differentiating once more and put $x = \pm 1$ and $y = z = 0$ for computing the Hessians, we get $\partial_y \partial_z x = 0$ and $\partial_y^2 x = \partial_z^2 x = \mp 1$. Hence the Hessians at the critical points are non-degenerate.
2. The gradient vector field is

   $$V(x, y, z) = (1, 0, 0) - \langle (1, 0, 0), (x, y, z) \rangle (x, y, z) = (1 - x^2, -xy, -xz).$$

   The integral curves are great arcs connecting $(-1, 0, 0)$ to $(1, 0, 0)$. To get their parametrized forms, we need to solve the equation

   $$x' = 1 - x^2, y' = -xy, z' = -xz$$

   with the boundary condition that $x(t \to -\infty) = -1, x(t \to \infty) = 1$, $y(t \to -\infty) = y(t \to \infty) = z(t \to -\infty) = z(t \to \infty) = 0$. The first equation gives

   $$x = \frac{e^{2\lambda t} - 1}{e^{2\lambda t} + 1}$$

   where λ can be taken to be any positive real constant (which just corresponds to scale of time). We fix $\lambda = 1$. Subsituting to the second and third equations, we get

   $$y = \frac{C_1 e^t}{1 + e^{2t}}, z = \frac{C_2 e^t}{1 + e^{2t}}.$$

   Since $x^2 + y^2 + z^2 = 1$, we get $C_1^2 + C_2^2 = 2$. Hence the solutions are

   $$(x, y, z) = \left(\frac{e^{2t} - 1}{e^{2t} + 1}, \frac{2e^t \cos\theta}{1 + e^{2t}}, \frac{2e^t \sin\theta}{1 + e^{2t}} \right)$$

   where θ is a real constant.

4. (RA)

Find a solution for the heat equation

$$\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0, \quad (t > 0, \quad 0 < x < 1),$$

with the initial condition $u(x,0) = A$ where A is a constant and the boundary conditions $u(0,t) = u(1,t) = 0, \quad t > 0$.

Solution: In view of the boundary conditions (Dirichlet), using linearity and separation of variables, we can write a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)e^{-n^2\pi^2 t}.$$

The coefficients B_n can be computed using a Fourier decomposition of the function $f(x) = u(x,0)$ given by the initial condition. A quick calculation ($B_n = 2\int_0^1 \sin(n\pi x)f(x)dx$) gives:

$$B_{2n} = 0 \quad B_{2n-1} = \frac{4A}{(2n-1)\pi}, \quad n = 1,2,3,\cdots$$

5. (AT)

(a) Show that a continuous map $f : X \to \mathbb{RP}^n$ factors through $S^n \to \mathbb{RP}^n$ if and only if the induced map $f^* : H^1(\mathbb{RP}^n;\mathbb{Z}/2) \to H^1(X,\mathbb{Z}/2)$ is zero.

(b) Show that a continuous map $f : X \to \mathbb{CP}^n$ factors through $S^{2n+1} \to \mathbb{CP}^n$ if and only if the induced map $f^* : H^2(\mathbb{CP}^n;\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is zero.

6. (CA) Let f be a meromorphic function on a contractible region $U \subset \mathbb{C}$, and let γ be a simple closed curve inside that region. Recall that the argument principle for a meromorphic function says that the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}$$

is equal to the number of zeroes minus the number of poles of f inside γ.

(a) Prove Rouché's Theorem. That is, assume (1) f and g are holomorphic in U, (2) γ is a simple, smooth, closed curve in U, and (3) $|f| > |g|$ on γ. Then the number of zeroes of $f + g$ inside γ is equal to the number of zeroes of f inside γ. You may assume the Argument Principle.

(b) Show that for any n, the roots of the polynomial

$$\sum_{i=0}^{n} z^i$$

all have absolute value less than 2.

Solution: (a) Apply the argument principle to $f + g$. Take note that the derivative of $\log(1 + \frac{g}{f})$ shows up. (b) Let $f = z^n$ and g be the summation of z^i from $i = 0$ to $n - 1$. Apply Rouché's theorem, noting that z^n has the same number of roots as $f + g$ (since they are polynomials of equal degree).

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Wednesday September 3, 2014 (Day 2)

1. (AT)

(a) Let X and Y be compact, oriented manifolds of the same dimension n. Define the *degree* of a continuous map $f : X \to Y$.

(b) What are all possible degrees of continuous maps $f : \mathbb{CP}^3 \to \mathbb{CP}^3$?

Solution: For the first part, the induced map $f^* : H^n(Y, \mathbb{Z}) \cong \mathbb{Z} \to H^n(X, \mathbb{Z}) \cong \mathbb{Z}$ is multiplication by some integer d; this is the degree of f.

For the second part, note that $H^*(\mathbb{CP}^3, \mathbb{Z}) \cong \mathbb{Z}[\zeta]/(\zeta^4)$ and that f^* is a ring homomorphism. If $f^*(\zeta) = m\zeta$, then $f^*(\zeta^3) = m^3\zeta^3$ and so the degree must be a cube. To see that all cubes occur, just consider the map $[X, Y, Z, W] \mapsto [X^m, Y^m, Z^m, W^m]$ for positive $d = m^3$; take complex conjugates to exhibit maps with negative degrees.

2. (A)

(a) Show that every finite extension of a finite field is simple (i.e., generated by attaching a single element).

(b) Fix a prime $p \geq 2$ and let $\mathbb{F}_p$ be the field of cardinality p. For any $n \geq 1$, show that any two fields of degree n over $\mathbb{F}_p$ are isomorphic as fields.

Solution: (a) If E/F is the field extension, then $E^\times$ is cyclic. Taking a generator x, we see that $E = F(x)$. (b) Any field extension of degree n is a splitting field for the polynomial $X^{p^n} - X$, hence is unique.

3. (CA) Fix two positive real numbers $a, b > 0$. Calculate the value of the integral

$$\int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx.$$

Solution: We compute the keyhole integral over a simple closed curve

$$C = C_r \cup C_R \cup [-R, -r] \cup [r, R],$$

where the closed intervals are on the y-axis of the complex plane. The curve C_r is a semicircle in the upper half-plane of radius $0 < r < R$, oriented so as to agree with the positive orientation on the real axis. Likewise C_R is on the upper half-plane. We integrate the function

$$F(z) = \frac{\exp(iaz) - \exp(ibz)}{z^2}.$$

In the interior of C, F has no singularities, so $\int_C F = 0$. Along C_r, we can use the power series expansion of exp to see that

$$\frac{\exp(iaz) - \exp(ibz)}{z^2} = \frac{1 - 1 + iaz - ibz + (iaz)^2/2 - (ibz)^2/2 + \dots}{z^2} = \frac{i(a-b)}{z} + h(z)$$

for some holomorphic function $h(z)$. So

$$\begin{aligned}
\lim_{r\to 0}\int_{C_r} F(z)dz &= \lim_{r\to 0}\int_{C_r}\frac{i(a-b)}{z}dz + \lim_{r\to 0}\int_{C_r} h(z)dz && (1)\\
&= \lim_{r\to 0}\int_{C_r}\frac{i(a-b)}{z}dz + \lim_{r\to 0} h(r) - h(-r) && (2)\\
&= \lim_{r\to 0}(a-b)\int_{\pi}^{0}\frac{i}{re^{it}}ire^{it}dt + 0 && (3)\\
&= (a-b)(-i^2)\pi && (4)\\
&= (a-b)\pi. && (5)
\end{aligned}$$

On the other hand, we utilize the following estimate as $R \to \infty$: Since $y > 0$ and $a > 0$, $e^{iaz} = e^{-ay}e^{ix}$ has a modulus less than 1. Likewise for e^{ibz}. This means that on C_R,

$$\left|\frac{\exp(iaz) - \exp(ibz)}{z^2}\right| \le \frac{2}{R^2}.$$

Hence the integral $\int_{C_R} F(z)dz$ is bounded by $2\pi/R$, which tends to zero as $R \to \infty$. So we obtain that

$$\begin{aligned}
0 &= \lim_{r\to 0, R\to\infty}\int_C F(z)dz && (6)\\
&= \lim_{r\to 0}\int_{C_r} Fdz + \lim_{R\to 0}\int_{C_R} Fdz + \int_{-\infty}^{+\infty} Fdz && (7)\\
&= \pi(a-b) + \int_{-\infty}^{+\infty} F(z)dz. && (8)
\end{aligned}$$

Looking at the real part of this equality we arrive at the conclusion that

$$\pi(b-a) = Real\int_{-\infty}^{+\infty} F(z)dz$$

which is what we seek.

4. (AG) Let $C \subset \mathbb{P}^2$ be the smooth plane curve of degree $d > 1$ defined by the homogeneous polynomial $F(X, Y, Z) = 0$

(a) If $p \in C$, find the homogeneous linear equation of the tangent line $\mathbb{T}_pC \subset \mathbb{P}^2$ to C at p.

(b) Let $\mathbb{P}^{2*}$ be the dual projective plane, whose points correspond to lines in $\mathbb{P}^2$. Show that the *Gauss map* $g : C \to \mathbb{P}^{2*}$ sending each point $p \in C$ to its tangent line $\mathbb{T}_pC \in \mathbb{P}^{2*}$ is a regular map.

(c) Let $C^* \subset \mathbb{P}^{2*}$ be the *dual curve* of C; that is, the image of the Gauss map. Assuming that the Gauss map is birational onto its image, what is the degree of $C^* \subset \mathbb{P}^{2*}$?

Solution: For the first part, the tangent line $\mathbb{T}_p C$ is given by the equation

$$\frac{\partial F}{\partial X}(p) \cdot X + \frac{\partial F}{\partial Y}(p) \cdot Y + \frac{\partial F}{\partial Z}(p) \cdot Z = 0.$$

For the second, the Gauss map is given by

$$g : p \mapsto \left[\frac{\partial F}{\partial X}(p), \frac{\partial F}{\partial Y}(p), \frac{\partial F}{\partial Z}(p) \right].$$

Since these have no common zeroes, the map is regular. For the third, since the partial derivatives of F are homogeneous of degree $d-1$, the preimage of a general line in $\mathbb{P}^{2*}$—that is, the zero locus of a general linear combination—will consist of $d(d-1)$ points (since the partials have no common zeroes, by Bertini a general linear combination will have only simple zeroes); thus $\deg(C^*) = d(d-1)$.

5. (DG) Let U the be upper half plane $U = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ and introduce the Poincaré metric

$$g = y^{-2}(dx \otimes dx + dy \otimes dy).$$

Write the geodesic equations.

Solution: A direct calculation gives $x'' - \frac{2}{y}x'y' = y'' - \frac{1}{y}[(x')^2 + 3(y')^2] = 0$.

6. (RA)

(a) Define what is meant by an *equicontinuous* sequence of functions on the closed interval $[-1,1] \subset \mathbb{R}$.

(b) Prove the Arzela-Ascoli theorem: that if $\{f_n\}_{n=1,2,\ldots}$ is a bounded, equicontinuous sequence of functions on $[-1,1]$, then there exists a continuous function f on $[-1,1]$ and an infinite subsequence $\Lambda \subset \{1,2,\ldots\}$ such that

$$\lim_{n \in \Lambda \text{ and } n \to \infty} \left(\sup_{t \in [-1,1]} |f_n(t) - f(t)| \right) = 0$$

.

Solution: First, a sequence $\{f_n\}$ of functions is *equicontinuous* if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that if $|t - t'| < \delta$ then $|f_n(t) - f_n(t')| < \epsilon$ for all n.

For the second part, here is a four-step proof:

1. We first show there exists a subsequence $\Lambda \subset \mathbb{N}$ such that $\forall r \in \mathbb{Q} \cap [-1,1]$, the sequence $\{f_n(r)\}_{n \in \Lambda}$ converges. We do this by first ordering $\mathbb{Q} \cap [-1,1]$, choosing a subsequence $\Lambda_1 \subset \mathbb{N}$ such that $\{f_n(r_1)\}_{n \in \Lambda_1}$ converges

(we can do this because $\{f_n\}$ is bounded); then choosing a subsequence $\Lambda_2 \subset \Lambda_1$ of that such that $\{f_n(r_2)\}_{n\in\Lambda_2}$ converges, and so on; we can do this such that if n_k is the smallest integer in Λ_k then $n_k \notin \Lambda_{k+1}$. We take $\Lambda = \{n_1, n_2, n_3, \dots\}$; since all but finitely many elements of Λ are in Λ_k, it follows that $\{f_n(r_k)\}_{n\in\Lambda}$ converges. Denote the limit $\lim_{n\in\Lambda} f_n(r_k)$ by $f(r_k)$.

2. Second, we claim that the function f on $\mathbb{Q}\cap[-1,1]$ defined in the first part satisfies the condition that $\forall\epsilon > 0$ there exists a $\delta > 0$ such that for $r, r' \in \mathbb{Q}\cap[-1,1]$, $|r - r'| < \delta \implies |f(r) - f(r')| < \epsilon$. This follows from the "up, over and down" argument: we have

$$|f(r) - f(r')| \leq |f(r) - f_n(r)| + |f_n(r) - f_n(r')| + |f_n(r') - f(r')|$$

and we can bound each term on the right by $\epsilon/3$ (the middle term by equicontinuity). It follows that for any $t \in [-1,1]$ and any sequence $\{q_1, q_2, \dots\} \subset \mathbb{Q}\cap[-1,1]$ converging to t, the sequence $f(q_n)$ is Cauchy; denote the limit by $f(t)$.

3. We claim that the function defined in the second part is continuous. This is again an up, over and down argument: if $t, t' \in [-1,1]$ and $r, r' \in \mathbb{Q}\cap[-1,1]$, we have

$$|f(t) - f(t')| \leq |f(t) - f(r)| + |f(r) - f(r')| + |f(r') - f(t')|$$

and again if we require t, t', r, r' to all lie in a sufficiently small interval we can bound each term by $\epsilon/3$.

4. We repeat the argument one more time. Choose N large, and consider the rational numbers $r \in \mathbb{Q}\cap[-1,1]$ with denominator N; that is, $\{k/N\}_{k=-N,-N+1,\dots,N}$. For any $t \in [-1,1]$, we choose $r = k/N$ close to t and write

$$|f_n(t) - f(t)| \leq |f_n(t) - f_n(r)| + |f_n(r) - f(r)| + |f(r) - f(t)|$$

and once more we can bound each term by $\epsilon/3$ by choosing n sufficiently large and r sufficiently close to t.

QUALIFYING EXAMINATION

Harvard University

Department of Mathematics

Thursday September 4, 2014 (Day 3)

1. (DG) The symplectic group $Sp(2n, \mathbb{R})$ is defined as the subgroup of $Gl(2n, \mathbb{R})$ that preserves the matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identify matrix. That is, it is composed of elements of $Gl(2n, \mathbb{R})$ that satisfy the relation

$$M^T \Omega M = \Omega.$$

(a) Show that every symplectic matrix is invertible with inverse $M^{-1} = \Omega^{-1} M^T \Omega$.

(b) Show that the square of the determinant of a symplectic metric is 1. (In fact, the determinant of a symplectic matrix is always 1, but you don't need to show this.)

(c) Compute the dimension of the symplectic group.

Solution: (a) Direct consequence from the definition since we can write $(\Omega^{-1} M^T \Omega) M = I_{2n}$ (b) Take the determinant on both side of the defining equation $M^T \Omega M = \Omega$ and use the fact that $\det M^T = \det M$. (c) Using the exponential map, describe the tangent space at the identity to be defined my the matrices m such that $m^T \Omega + \Omega m = 0$ which can also be written as $\Omega m^T \Omega = m$. Writing $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in terms of $n \times n$ blocks (a, b, c and d) and deduce the condition that these blocks have to satisfy ($d = -a^T, b^T = b, c^T = c$) to have $\Omega m^T \Omega = m$. It follows that the dimension is $n(2n+1)$.

2. (RA) Suppose that σ is a positive number and f is a non-negative function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} f(x)dx = 1; \quad \int_{\mathbb{R}} x f(x) dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} x^2 f(x) dx = \sigma^2.$$

Let $\mathcal{P}$ denote the probability measure on $\mathbb{R}$ with density function f.

(a) Supposing that ρ is a positive number, give a non-trivial upper bound in terms of σ for the probability as measured by $\mathcal{P}$ of the subset $[\rho, \infty)$.

(b) Given a positive integer N, let $\{X_1, \ldots, X_N\}$ denote N independent random variables on $\mathbb{R}$, each with the same probability measure $\mathcal{P}$. Let S_N be the random variable on $\mathbb{R}^N$ given by

$$S_N = \frac{1}{N}\sum_{i=1}^{N} X_i$$

What are the mean and standard deviation of S_N?

(c) Let $\{X_1, X_2, \ldots, X_N\}$ be independent random variables on $\mathbb{R}$, each with the same probability measure $\mathcal{P}$, and let $P_N(x)$ denote the function on $\mathbb{R}$ given by the probability that

$$\frac{1}{\sqrt{N}}\sum_{k=1}^{N} X_k \ < \ x.$$

Given $x \in \mathbb{R}$, what is the limit as $N \to \infty$ of the sequence $\{P_N(x)\}$?

Solution: For the first part, the probability assigned to the interval is $\int_\rho^\infty f(x)dx$. An upper bound is derived by noting that the probability is no greater than $\int_\rho^\infty \frac{x^2}{\rho^2} f(x)dx$ and this in turn is at most $\frac{\sigma^2}{\rho^2}$ This is Chebyshevs inequality.

For the second part, the mean is 0 and the standard deviation is $\frac{1}{\sqrt{N}}\sigma$.

Finally, the central limit theorem says that

$$\lim_{N\to\infty} P_N(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}.$$

3. (AG) Let X be the blow-up of $\mathbb{P}^2$ at a point.

(a) Show that the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and X are all birational.

(b) Prove that no two of the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and X are isomorphic.

Solution: For the first part, we can simply observe that all three surfaces contain the affine plane $\mathbb{A}^2$ as a Zariski open subset.

For the second, there are many invariants that we can use to distinguish $\mathbb{P}^2$ from $\mathbb{P}^1 \times \mathbb{P}^1$: the topological Euler characteristic; the self-intersection of the canonical bundle, or the rank of the Picard group all work. To see that X is not isomorphic to either, note that X contains a curve of negative self-intersection (the exceptional divisor), while $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ do not.

4. (AT) Suppose that G is a finite group whose abelianization is trivial. Suppose also that G acts freely on S^3. Compute the homology groups (with integer coefficients) of the orbit space $M = S^3/G$.

Solution: Note that M is a smooth manifold, and that $\pi_1 M = G$. By Poincare's theorem $H_1 S^3/G = 0$, as is $H^1(S^3/G; A) = \hom(\pi_1 M, A)$ for any abelian group A. This implies that M is orientable. It then follows from Poincare duality that $H_2(M; A) = 0$ for any abelian group A and that $H_3(M; A) = A$.

5. (CA) Recall that a function $u : \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic* if $\Delta u := \partial_x^2 u + \partial_y^2 u = 0$. Prove the following statements using harmonic conjugates and standard complex analysis.

(a) Show that the average value of a harmonic function along a circle is equal to the value of the harmonic function at the center of the circle.

(b) Show that the maximum value of a harmonic function on a closed disk occurs only on the boundary, unless u is constant.

Solution: Cauchy Integral Formula, and maximum principle. In detail:

(a) Let v be the harmonic conjugate for u so $u+iv = f$ is an analytic function on $\mathbb{R}^2$. By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

for any point a and any closed, simple curve surrounding a. Taking γ to be a circle of radius R centered at a, and parametrizing $z(t) = Re^{it} + a$, we thus have

$$u(a) + iv(a) = \frac{1}{2\pi i}\int_\gamma \frac{u+iv}{Re^{it}} iRe^{it}dt.$$

Equating real and imaginary parts, we obtain

$$u(a) = \frac{1}{2\pi R}\int_\gamma udt.$$

(b) If u is harmonic, let v be a harmonic conjugate so $f = u+iv$ is holomorphic. By the maximum modulus principle, $|e^f|$ must obtain a maximum only along the boundary (unless f is constant). But $|e^f| = |e^u| = e^u$, and since exp is strictly monotone and continuous, e^u obtains a maximum if and only if u does.

6. (A) Let G be a finite group.

(a) Let V be any $\mathbb{C}$-representation of G. Show that V admits a Hermitian, G-invariant inner product.

(b) Let N be a $\mathbb{C}[G]$-module which is finite-dimensional over $\mathbb{C}$, and let $M \subset N$ a submodule. Show that the inclusion splits.

(c) Consider the action of S_3 on $\mathbb{C}^3$ given by permuting the axes. Decompose $\mathbb{C}^3$ into irreducible S_3-representations.

Solution: (a) Put an arbitrary inner product $\langle , \rangle$ on V, then define

$$(v, w) := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle.$$

(b) Take the orthogonal complement to M under a G-invariant inner product.
(c) Clearly the diagonal $z_1 = z_2 = z_3$ is an invariant subspace. By using the methods above (or by writing an invariant linear equation) we deduce that an orthogonal complement is given by the plane $z_1 + z_2 + z_3 = 0$. This two-dimensional representation has no subrepresentations, so must be the unique 2-dimensional irreducible representation.