QUALIFYING EXAMINATION

HARVARD UNIVERSITY

Department of Mathematics

Tuesday September 16 2008 (Day 1)

1. (a) Prove that the Galois group G of the polynomial $X^6 + 3$ over $\mathbb{Q}$ is of order 6.

(b) Show that in fact G is isomorphic to the symmetric group S_3.

(c) Is there a prime number p such that $X^6 + 3$ is irreducible over the finite field of order p?

2. Evaluate the integral

$$\int_0^\infty \frac{\sqrt{t}}{(1+t)^2} dt.$$

3. For $X \subset \mathbb{R}^3$ a smooth oriented surface, we define the *Gauss map* $g : X \to S^2$ to be the map sending each point $p \in X$ to the unit normal vector to X at p. We say that a point $p \in X$ is *parabolic* if the differential $dg_p : T_p(X) \to T_{g(p)}(S^2)$ of the map g at p is singular.

(a) Find an example of a surface X such that every point of X is parabolic.

(b) Suppose now that the locus of parabolic points is a smooth curve $C \subset X$, and that at every point $p \in C$ the tangent line $T_p(C) \subset T_p(X)$ coincides with the kernel of the map dg_p. Show that C is a planar curve, that is, each connected component lies entirely in some plane in $\mathbb{R}^3$.

4. Let $X = (S^1 \times S^1) \setminus \{p\}$ be a once-punctured torus.

(a) How many connected, 3-sheeted covering spaces $f : Y \to X$ are there?

(b) Show that for any of these covering spaces, Y is either a 3-times punctured torus or a once-punctured surface of genus 2.

5. Let X be a complete metric space with metric ρ. A map $f : X \to X$ is said to be *contracting* if for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < \rho(x, y).$$

The map f is said to be *uniformly contracting* if there exists a constant $c < 1$ such that for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < c \cdot \rho(x, y).$$

(a) Suppose that f is uniformly contracting. Prove that there exists a unique point $x \in X$ such that $f(x) = x$.

(b) Give an example of a contracting map $f : [0, \infty) \to [0, \infty)$ such that $f(x) \neq x$ for all $x \in [0, \infty)$.

6. Let K be an algebraically closed field of characteristic other than 2, and let $Q \subset \mathbb{P}^3$ be the surface defined by the equation

$$X^2 + Y^2 + Z^2 + W^2 = 0.$$

(a) Find equations of all lines $L \subset \mathbb{P}^3$ contained in Q.

(b) Let $\mathbb{G} = \mathbb{G}(1, 3) \subset \mathbb{P}^5$ be the Grassmannian of lines in $\mathbb{P}^3$, and $F \subset \mathbb{G}$ the set of lines contained in Q. Show that $F \subset \mathbb{G}$ is a closed subvariety.

QUALIFYING EXAMINATION

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Wednesday September 17 2008 (Day 2)

1. (a) Show that the ring $\mathbb{Z}[i]$ is Euclidean.

(b) What are the units in $\mathbb{Z}[i]$?

(c) What are the primes in $\mathbb{Z}[i]$?

(d) Factorize $11 + 7i$ into primes in $\mathbb{Z}[i]$.

2. Let $U \subset \mathbb{C}$ be the open region

$$U \ = \ \{z : |z-1| < 1 \text{ and } |z-i| < 1\}\,.$$

Find a conformal map $f : U \to \Delta$ of U onto the unit disc $\Delta = \{z : |z| < 1\}$.

3. Let n be a positive integer, A a symmetric $n \times n$ matrix and Q the quadratic form

$$Q(x) = \sum_{1 \le i,j \le n} A_{i,j} x_i x_j.$$

Define a metric on $\mathbb{R}^n$ using the line element whose square is

$$ds^2 = e^{Q(x)} \sum_{1 \le i \le n} dx^i \otimes dx^i.$$

(a) Write down the differential equation satisfied by the geodesics of this metric

(b) Write down the Riemannian curvature tensor of this metric at the origin in $\mathbb{R}^n$.

4. Let H be a separable Hilbert space and $b : H \to H$ a bounded linear operator.

(a) Prove that there exists $r > 0$ such that $b + r$ is invertible.

(b) Suppose that H is infinite dimensional and that b is compact. Prove that b is not invertible.

5. Let $X \subset \mathbb{P}^n$ be a projective variety.

(a) Define the *Hilbert function* $h_X(m)$ and the *Hilbert polynomial* $p_X(m)$ of X.

(b) What is the significance of the degree of p_X? Of the coefficient of its leading term?

(c) For each m, give an example of a variety $X \subset \mathbb{P}^n$ such that $h_X(m) \neq p_X(m)$.

6. Let $X = S^2 \vee \mathbb{RP}^2$ be the wedge of the 2-sphere and the real projective plane. (This is the space obtained from the disjoint union of the 2-sphere and the real projective plane by the equivalence relation that identifies a given point in S^2 with a given point in $\mathbb{RP}^2$, with the quotient topology.)

(a) Find the homology groups $H_n(X, \mathbb{Z})$ for all n.

(b) Describe the universal covering space of X.

(c) Find the fundamental group $\pi_1(X)$.

QUALIFYING EXAMINATION

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Thursday January 31 2008 (Day 3)

1. For $z \in \mathbb{C} \setminus \mathbb{Z}$, set

$$f(z) = \lim_{N \to \infty} \left(\sum_{n=-N}^{N} \frac{1}{z+n} \right)$$

(a) Show that this limit exists, and that the function f defined in this way is meromorphic.

(b) Show that $f(z) = \pi \cot \pi z$.

2. Let p be an odd prime.

(a) What is the order of $GL_2(\mathbb{F}_p)$?

(b) Classify the finite groups of order p^2.

(c) Classify the finite groups G of order p^3 such that every element has order p.

3. Let X and Y be compact, connected, oriented 3-manifolds, with

$$\pi_1(X) = (\mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \qquad \text{and} \qquad \pi_1(Y) = (\mathbb{Z}/6\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

(a) Find $H_n(X, \mathbb{Z})$ and $H_n(Y, \mathbb{Z})$ for all n.

(b) Find $H_n(X \times Y, \mathbb{Q})$ for all n.

4. Let $\mathcal{C}_c^\infty(\mathbb{R})$ be the space of differentiable functions on $\mathbb{R}$ with compact support, and let $L^1(\mathbb{R})$ be the completion of $\mathcal{C}_c^\infty(\mathbb{R})$ with respect to the L^1 norm. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{h \to 0} \frac{1}{h} \int_{|y-x|<h} |f(y) - f(x)| dy = 0$$

for almost every x.

5. Let $\mathbb{P}^5$ be the projective space of homogeneous quadratic polynomials $F(X, Y, Z)$ over $\mathbb{C}$, and let $\Phi \subset \mathbb{P}^5$ be the set of those polynomials that are products of linear factors. Similarly, let $\mathbb{P}^9$ be the projective space of homogeneous cubic polynomials $F(X, Y, Z)$, and let $\Psi \subset \mathbb{P}^9$ be the set of those polynomials that are products of linear factors.

(a) Show that $\Phi \subset \mathbb{P}^5$ and $\Psi \subset \mathbb{P}^9$ are closed subvarieties.

(b) Find the dimensions of Φ and Ψ.

(c) Find the degrees of Φ and Ψ.

6. Realize S^1 as the quotient $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and consider the following two line bundles over S^1:

L is the subbundle of $S^1 \times \mathbb{R}^2$ given by

$$L = \{(\theta, (x, y)) : \cos(\theta) \cdot x + \sin(\theta) \cdot y = 0\}; \text{and}$$

M is the subbundle of $S^1 \times \mathbb{R}^2$ given by

$$M = \{(\theta, (x, y)) : \cos(\theta/2) \cdot x + \sin(\theta/2) \cdot y = 0\}.$$

(You should verify for yourself that M is well-defined.) Which of the following are trivial as vector bundles on S^1?

(a) L

(b) M

(c) $L \oplus M$

(d) $M \oplus M$

(e) $M \otimes M$