Lecture 3 — p-adic interpolation of the algebraic representations of $\Gamma_0(p)$

(next lecture, application to the construction of unitary eigenvarieties)

Notations $\quad G = GL_n(\mathbb{Q}_p)$, $\quad L, B$ = lower and upper Borel coeff. in $\mathbb{Q}_p$.

$n \geq 1$ integer $\quad N \subset B$ unipotent radical, $\quad T$ diagonal torus.

$K = GL_n(\mathbb{Z}_p)$, $\quad I = (\Gamma_0(p)) = \{ g \in K,\ g \equiv \begin{pmatrix} x & * \\ 0 & x \end{pmatrix} \bmod p \}$

$U^+ = \{ (p^{a_1}, \dots, p^{a_n}) \in T,\ a_1 \leq \dots \leq a_n \}$, $\quad M = \langle I, U^+ \rangle$ monoid generated

$\cup$

U^{++} ———————— $a_1 < \dots < a_n$

Ⓐ Case $n = 2$: an example (Following Buzzard, Stevens)

$k \geq 0$ integer, $\quad V_k := \mathrm{Symm}^k\, \mathbb{Q}_p^2 \circlearrowleft G$, irreducible

$$\mathbb{Q}_p^2 = \mathbb{Q}_p x \oplus \mathbb{Q}_p y$$

$$V_k = \langle x^k, x^{k-1}y, \dots, y^{k-1}x, y^k \rangle_{\mathbb{Q}_p} = x^k \langle 1, t, \dots, t^k \rangle_{\mathbb{Q}_p}, \quad t := \frac{y}{x}$$

Induced action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\mathbb{Q}_p[t]_{\leq k}$: $\quad \gamma(P(t)) = j(\gamma)^k\, P\left(\frac{b + dt}{a + ct}\right)$

where $\quad j(\gamma) = a + ct$

k-twist $\qquad$ extends to $\mathbb{Q}_p(t)$ — fixed

rational fct / $\mathbb{Q}_p$

Geometric picture $\quad X = \mathbb{P}^1_{/\mathbb{Q}_p}$ / $\mathcal{O}(1)$ line bundle, $\quad$ "$(x : y) = (1 : t)$"

$G = \mathrm{Aut}_{\mathbb{Q}_p}(X)$, $\quad V_k = H^0(X, \mathcal{O}(k)) \hookrightarrow H^0(X_\eta, \mathcal{O}(k))$

gen pt.

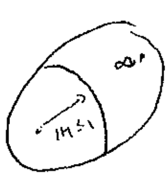
NB: wk for any field, not only $\mathbb{Q}_p$, $\quad H^0(X_\eta, \mathcal{O}(k)) \simeq \mathbb{Q}_p(t) \circlearrowleft$ k-tw. action $\quad \forall k \in \mathbb{Z}$.

Main observations

- M preserves $\mathcal{F} = \{ t, |t| \leq 1 \} \subset \mathbb{A}^1 \subset X$ (disc)

  because $\quad |a + ct| = |a| = 1$
  if $p \mid c$, $t \in \mathcal{F}$, $\gamma \in K$, $\qquad (1, p) \cdot t = pt$.

- $\mathcal{O}(1)$ is trivial on $\mathcal{F}$ (x is a generator)



quences

i) We get a continuous rep of I on $\mathcal{O}(\mathcal{F}) \simeq \mathbb{Q}_p\langle t\rangle$ (gauss norm) by the same formulas, it extends to M, and $u := (1, p)$ satisfies $u(t) = pt$ $(u(\sum a_n t^n) = \sum a_n p^n t^n)$ hence is compact.

ii) $\forall k \in \mathbb{Z}$, $\mathcal{V}_k := H^0(\mathcal{F}, \mathcal{O}(k)) = x^k \mathcal{O}(\mathcal{F}) \simeq \mathcal{O}(\mathcal{F}) \curvearrowleft$ M action twisted by j^k

$x^k f \leftarrow f$

$\mathcal{V}_k \supset V_k$ if $k \geq 0$.

And we can "recover V_k": if $\varphi \in \mathbb{Q}_p[IuI]$, $0 \neq v \in \mathcal{V}_k$ $(k \geq 0)$ eigen vector $\varphi(v) = \lambda v$ with $v(\lambda) < k+1$, then v analytically continue to X. (the norm of u on $\mathcal{V}_k / V_k$ is $\leq |p^{k+1}|$, clear)

iii) If $\chi : \mathbb{Z}_p^* \to L^*$ a continuous character, $\mathcal{V}_\chi$ makes sense:

$L \otimes_{\mathbb{Q}_p} \mathcal{O}(\mathcal{F}) \curvearrowleft \chi(j)$ - twisted action

$\longrightarrow$ a pretty nice family of rep of I parameterized by W.

(B) the big Bruhat - Iwahori cell

$X = L \backslash G$ flag variety, look at $\curvearrowleft I$ on the right.

Lemma i) $G = \coprod_{w \in \mathfrak{S}_n} LwI$

ii) $L\backslash L.I \subset L\backslash L.B$ (big Bruhat cell (open sub. sh.)) so $L\backslash LI \simeq N(\mathbb{Z}_p)$ is the $\mathbb{Q}_p$-points of an affinoid subdomain of X^{an} call it $\mathcal{F}$ ("big B.I cell")

$$\begin{array}{ccc} L\backslash L.I & \subset & L\backslash L.B \\ \uparrow s & & \updownarrow s \\ N(\mathbb{Z}_p) & \subset & N \simeq \mathbb{A}^{\frac{n(n-1)}{2}} \end{array}$$

$\mathcal{F} \simeq$ $\frac{n(n-1)}{2}$-dim closed polydisc in $\mathbb{A}^{\frac{n(n-1)}{2}}$

iii) $\mathcal{F}$ is stable under M.

If $u \in U^{++}$, $u.\underbrace{\mathcal{F}(\mathbb{Q}_p)}_{N(\mathbb{Z}_p)} \subset p\underbrace{\mathcal{F}(\mathbb{Q}_p)}_{N(p\mathbb{Z}_p)}$.

(precisely: $\mathcal{F} \subset \mathbb{A}^{\frac{n(n-1)}{2}} \simeq (n_{ij})$ ... $|k_{ij}| \leq 1$ ...

i) Use $G = LK$ and Bruhat dec. of $GL_n(\mathbb{F}_p)$

ii) - Lower - Upper dec. shows $L \backslash LB \simeq N$ open subscheme of X

(defined by $\Delta_i \neq 0 \quad \forall i \in \{1 \dots n\}$.)

$\uparrow$ ith principal minor

- Use $I \simeq (L \cap I) \times (N \cap I)$ (Iwahori - decomp.)

iii) - I stable by definition

- $\begin{pmatrix} 1 & & n_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} (p^{a_i}) \sim \begin{pmatrix} 1 & & n_{ij} p^{a_j - a_i} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in N(\mathbb{Z}_p)$ if $a_j \geq a_i, \forall j > i$

$\quad\;\; \underset{N(\mathbb{Z}_p)}{\cap}$ $\qquad\qquad\qquad\qquad\qquad\qquad p N(\mathbb{Z}_p)$ if $a_j > a_i, \forall j > i$

<u>Examples</u> - $n = 2$, $X = \mathbb{P}^1$, $N(\mathbb{Z}_p) \simeq \mathbb{Z}_p$, we recover the preceding case

- $n = 3$, action of I on ~~F~~ a little more complicated already.

A simpler way is to view $\mathcal{F} \subset \mathbb{P}(V^*) \times \mathbb{P}(\Lambda^2 V^*)$ $\quad (V = \mathbb{Q}_p^3)$

we get $\mathcal{O}(\mathcal{F}) \simeq \mathbb{Q}_p \dfrac{\langle u, v, x, y \rangle}{(y - ux + v)}$ incidence relation, also $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 \Lambda^2 V \supset$ <u>line</u>

<u>Consequences</u> Set $\mathcal{O}(\mathcal{F}, r) = \mathbb{Q}_p$ - algebra functions $\underset{\overset{\shortparallel}{N(\mathbb{Z}_p)}}{\mathcal{F}(\mathbb{Q}_p)} \longrightarrow \mathbb{Q}_p$.

$r \geq 0$ integer

analytic on each "$a + p^r N(\mathbb{Z}_p)$"

+ sup norm on $\coprod_{a \in N(\mathbb{Z}/p^r\mathbb{Z})} \mathrm{disc}(a, p^{-r})$.

<u>Then</u> $\mathcal{O}(\mathcal{F}, r)$ is a p-adic representation of M, I acts by isometries and $u \in U^{++}$ by compact. op. of norm ≤ 1

$u \in U^{++}, r \geq 1$, $\exists$. fact.

$$\begin{tikzcd} \mathcal{O}(\mathcal{F}, r) \arrow[rr, "u"] \arrow[dr] & & \mathcal{O}(\mathcal{F}, r) \\ & \mathcal{O}(\mathcal{F}, r-1) \arrow[ur, "\text{res. (compact.)}"'] & \end{tikzcd}$$

Weights

1) Integral weights

$\mathbb{Z}^{n,+} = \{ (k_1, \dots, k_n) \in \mathbb{Z}^n,\ k_1 \geq \dots \geq k_n \}$ (dim 1 over $\mathbb{Q}_p$)

$\forall k \in \mathbb{Z}^{n,+}$, $\exists !$ irreducible alg. rep. of G V_k such that $V_k^N \circlearrowleft T$ acts through $(x_1 \cdots x_n) \longmapsto x_1^{k_1} \cdots x_n^{k_n}$

fundamental rep. $\Lambda^i V = V_{(\underbrace{1, \dots, 1}_{i}, 0, \dots, 0)}$

tautologically, $X \xrightarrow{\pi_i} \mathbb{P}(\Lambda^i V^*)$, $\pi_i^* \mathcal{O}(1) =: \mathcal{L}_i$ line bundle s.t. $H^0(X, \mathcal{L}_i) = \Lambda^i V$.

Lemma

i) (Borel-Weil-Bott) $k \in \mathbb{Z}^{n,+}$, $V_k \simeq H^0(X, \mathcal{L}_1^{k_1 - k_2} \otimes \dots \otimes \mathcal{L}_{n-1}^{k_{n-1} - k_n}) \otimes \det V^{k_n}$

ii) $V_k \longhookrightarrow \mathcal{V}_k := H^0(\mathcal{F}, \cancel{\mathcal{O}_{\mathcal{F}}} \quad) \otimes \det V^{k_n}$

iii) each $\mathcal{L}_i$ is trivial on $\mathcal{F}$ (and trivialized (over $\mathbb{Z}_p$) by Δ_i)

iv) if $\varphi \in \mathbb{Q}_p[I u I]$, $u = (1, p, \dots, p^{n-1})$, $0 \neq w \in \mathcal{V}_k$ ($k \in \mathbb{Z}^{n,+}$) is an eigenvector $\varphi(w) = \lambda w$ with $v(\lambda) < k_{i+1} - k_i + 1$ for all i, then $w \in V_k$.

Pf: i) Well known

ii) $\mathcal{F}$ is open and X is irreducible

iii) $\mathcal{L}_i$ is trivial on $L \backslash LB$ ($\simeq \mathbb{A}^{\frac{n(n-1)}{2}}$), $\mathcal{O}(1)$ on $\mathbb{P}(\Lambda^i V^*)$ trivial on $\Delta_i \neq 0$.

iv)

$$0 \to V_k \hookrightarrow \mathcal{V}_k \longrightarrow \mathcal{V}_k / V_k \longrightarrow 0$$

use Plücker emb. $X \hookrightarrow \prod_i \mathbb{P}(\Lambda^i V^*)$

compute norm of $(1, p, \dots, p^{n-1})$ here

$$0 \to \left(\bigotimes_{i=1}^{n-1} \mathrm{Symm}^{k_i - k_{i+1}} \Lambda^i V \right) \otimes \det \Lambda^{k_n} \to \bigotimes_{i=1}^{n-1} H^0\left(\mathbb{P}(\Lambda^i V^*)^{|\Delta_i| \leq 1}, \mathcal{O}(k_i - k_{i+1}) \right) \otimes \det \Lambda^{k_n} \to \text{quotient} \to$$

and we compute the norm here

easy

$= H^0(\mathbb{P}(\Lambda^i V^*), \mathcal{O}(k_i - k_{i+1}))$

p-adic weights

Cocycles Set $j_n := \det_{|I}$. By ① iii), we have $n-1$ 1-cocycles $I \to \mathcal{O}(\mathcal{F})^\times$ (even on $\mathbb{Z}_p$)
defined by $j_i(\gamma) = \dfrac{\gamma(\Delta_i)}{\Delta_i}$ ("first row of γ acting on $\wedge^i V$")
looks like $a+ct$

Fact: Each j_i extends to a 1-cocycle on $M \longrightarrow \mathcal{O}(\mathcal{F})^*$ s.that
$j_i(U^+) = 1$.

Pf: Use that $M = \coprod_{u \in U^+} I u I$, $I u I u' I \subset I u u' I$ to twist the natural j_i's by a character.

- Let $\mathcal{W} = \mathrm{Hom}_{\text{gr. cont}}(T(\mathbb{Z}_p), \mathbb{G}_m^{\text{rig}})$ p-adic character space

fix $\Omega \subset \mathcal{W}$ open affinoid, universal character $\chi^{\text{univ}} : T(\mathbb{Z}_p) \longrightarrow \mathcal{O}(\Omega)^\times$
whose restriction to $(1+p^r\mathbb{Z}_p)^n$ is analytic for some $r \geq 0$

define $\mathcal{V}_{\Omega,r} := \mathcal{O}(\mathcal{F}_r)\hat{\otimes}_{\mathbb{Q}_p}\mathcal{O}(\Omega)$ as M-module ($\mathcal{O}(\Omega)$-linear)
twisted by $\chi_1/\chi_2(j_1)\ \chi_2/\chi_3(j_2) \cdots \chi_{n-1}/\chi_n(j_{n-1})\ \chi_n(j_n)$

Note that if $\chi : \mathbb{Z}_p^\times \to A^\times$ is analytic on $1+p^r\mathbb{Z}_p$, then

$\forall \gamma \in I$, $\chi(j_i(\gamma)) : \mathcal{F}(\mathbb{Q}_p) \longrightarrow \mathbb{Z}_p^* \xrightarrow{\chi} A^\times$
$i=1,\dots,n-1$
is analytic on the r-thickening, i.e. $\in (\mathcal{O}(\mathcal{F},r)\hat{\otimes} A)^\times$.

Prop
- $\mathcal{V}_{\Omega,r}$ is an ON-able $\mathcal{O}(\Omega)$-module
- I acts continuously, by isometries.
- U^{++} acts through (constant) compact $\mathcal{O}(\Omega)$-endomorphisms of norm ≤ 1.

Pf: clear.