Lecture ② The trianguline deformation functor of a refined crystalline rep'n (with J. Bellaïche)

(G. Chenevier March 13)

Recall $L/\mathbb{Q}_p$ finite coeff extension, $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, R_L = Robba ring of $\mathbb{Q}_p$ coeff. in L

We say D is triangular if $\exists$ $D_0 = \{0\} \subsetneq D_1 \subsetneq \cdots \subsetneq D_{d-1} \subsetneq D_d = D$ sub(φ,Γ)modules

D_i has rk i saturated (= direct summand)

We often write $\mathrm{Fil}_i D = D_i$

Parameters of the triangulation: $D_i/D_{i-1} \simeq R_L(\delta_i)$ $\quad \delta_i : \mathbb{Q}_p^\times \to L^*$ cont. char. $i = 1 \cdots d$

Say V is trianguline if $D_{rig}(V)$ is triangular.

Ⓐ Triangulations of crystalline rep.

Let V be a d-dim L-rep. of $G_{\mathbb{Q}_p}$ which is crystalline (or a crystalline (φ,Γ)-module)

$D_{crys}(V)$ $\quad$ d-dim. L-v. space + φ + wt filtration $(\mathrm{Fil}^i)_{i\in\mathbb{Z}}$.

(*) Assume that charpoly(φ) split in $L[T]$.

Def (Mazur) A refinement is the datum of a φ-stable complete flag

$\mathcal{F}_0 = \{0\} \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_{d-1} \subsetneq \mathcal{F}_d = D_{crys}(V)$ $\quad$ $\mathcal{F}_i$ φ-stable i-dim L-sub v.sp

By (*) there are always (many) refinements.

Let $\mathcal{F} = (\mathcal{F}_i)$ be a refinement, it determines 2 orderings.

$\mathcal{F}$ (i) An ordering of the eigenvalues of φ defined by

$$\det(T - \varphi|_{\mathcal{F}_j}) = \prod_{i=1}^{j} (T - \varphi_i)$$

(this in turn determines $\mathcal{F}$ if the φ_i's are distinct)

$\mathcal{F}$ (ii) An ordering of the HTW of V, $s_1, \cdots s_d$, defined by

$$WT(\mathcal{F}_j) = (s_1, \cdots, s_j) \qquad (\mathcal{F}_j \subset \mathcal{F}_{j+1})$$

Basic construction:

$$\mathcal{F}=(\mathcal{F}_i) \longmapsto \mathrm{Fil}_i D = \mathcal{R}_L[\tfrac{1}{t}]\mathcal{F}_i \cap D \quad , \boxed{D = D_{\mathrm{rig}}(V)}$$

sub (φ,Γ)-module rk i
direct summand (by Colmez's lemma and class. in rk 1

Proposition
(i) $(\mathcal{F}_i) \mapsto (\mathrm{Fil}_i D)$ is a bijection between {refinements of V} and {triangulations of D}, whose inverse is $(D_i) \mapsto (D_i[\tfrac{1}{t}]^\Gamma)$

(ii) In this bijection, the parameter δ_i of $(\mathrm{Fil}_i D)$ is given by
$$\delta_i(p) = p^{-s_i}\varphi_i \ , \ \delta_{i|\Gamma} = \chi^{-s_i}$$
where φ_i and s_i have been defined before.

proof: (i) $\mathcal{F}_1 = L\cdot v$, $\varphi v = \lambda v$.
$\cap$
$D[\tfrac{1}{t}]^\Gamma$ by Berger, so $\exists s \in \mathbb{Z}$, $\mathcal{R}_L t^{-s} v \subset D$.

By Colmez we can assume that saturated, hence we have an ex.

$$0 \to \mathcal{R}_L t^{-s} v \to D \to D/\mathcal{R}_L t^{-s} v \to 0$$

$\mathcal{R}_L t^{-s} v = \mathcal{R}_L(\delta)$; $D/\mathcal{R}_L t^{-s} v$: f.t + torsion free = free

$\left| \begin{array}{l} \delta(p) = p^{-s}\varphi_1 \\ \delta_{|\Gamma} = \chi^{-s} \end{array}\right.$, and we continue by induction.

(ii) harder, identify the s above, the main lemma is then the following.

Lemma D a (φ,Γ)-module over $\mathcal{R}_L$, $\lambda \in L^\times$, $v \in D_{\mathrm{rig}}(D)^{\varphi=\lambda}$,
then $v \in \mathrm{Fil}^i D_{\mathrm{rig}}(D) \Leftrightarrow v \in t^i D$.

Pf: $\Leftarrow$ obvious, (and doesn't use the assumption.) we show $\Rightarrow$.
By Berger, $p^n(p-1) \geq r_n \gg r(D)$, $\varphi^n(v) \in \left(t^i D_r \otimes_{\mathcal{R}_r} K_n[[t]]\right)^\Gamma$
$v \in D_r[\tfrac{1}{t}]$; $\varphi^n(v) = \lambda^n v$.

This shows that $v \in t^i D_{r_n} \otimes k_n[[t]]$, $\forall n \gg 0 \Rightarrow v \in t^i D_r$ (3)/5

Rk. * In part., crystalline rep. are trianguline, in $d!$ ways generically.

* exercise (Colmez thm.) $d=2$, D is triangular iff

$\exists\, \delta$ and $\lambda \in L^*$, $D_{\mathrm{crys}}(D \otimes R(\delta))^{\varphi=\lambda} \neq 0$.

(B) Non critical refinements

Assume that HTW of V are distinct, say $k_1 < \dots < k_d$.
Let $(\mathcal{F}_i)$ be a ref. of V.

Def $(\mathcal{F}_i)$ is non critical if $\mathcal{F}_i \oplus \mathrm{Fil}^{k_{i+1}} D_{\mathrm{crys}}(V) = D_{\mathrm{crys}}(V)$.

This is the generic situation; it is equivalent to ask that $s_i = k_i$ $\forall i$ By prop (ii).

Ex: . $d=2$, $D = D_{\mathrm{rig}} V$, V HTW $\{0, k-1\}$ $k>1$
All ref. are critical except when V is ordinary split $\simeq \mathbb{Q}_p \oplus \mathbb{Q}_p(k-1)$
and $\mathcal{F}_1 = D_{\mathrm{crys}}(\mathbb{Q}_p(k-1))$.

Rk • numerical conditions can imply non criticalness using wk admissibility.
• Weakly ref. V have a nice deformation theory.

(C) Deformation theory; the trianguline def. functor

Goal: choose V as before, fix an $\mathcal{F} \Longrightarrow$ triangulation of D.
We will deform D with this triangulation.

A: f.dim. local $\mathbb{Q}_p$ algebra $+ A/m_A \xrightarrow{\sim} L$, category $\mathcal{C}$, $R_A = R_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} A$
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lemma. • D_{rig} induces a bijection $\left\{ V_A \begin{array}{l} A\text{-linear cont.} \\ \text{rep. of } G_{\mathbb{Q}_p} \\ \text{free of rk } d/A \end{array} \right\} \xrightarrow{\sim} \left\{ D_A, \begin{array}{l} \varphi\Gamma\text{-modules} \\ + \text{action of } A \\ \text{free of rk } d \\ \text{over } R_A \end{array} \right\}$

$+ V_A \otimes_A L \xrightarrow{\sim} V$ $\qquad + D_A \otimes_A L \xrightarrow{\sim} D$

$\mathcal{X}_{\dots}(A) := \{\dots$

II: Main fact, easy conseq. of Kedlaya's work,
"An extension between two étale (φ,Γ)-modules is again étale".

Def A $(\varphi\Gamma)$-module D_A is triangular if $\exists$ $D_0 \subsetneq \dots \subsetneq D_i \subsetneq \subsetneq D_d$ free rk over R_A D_i $(\varphi\Gamma)$-R_A-submodule free rk i over R_A direct summand.

Again $D_i/_{D_{i-1}} \simeq R_A(\delta_i)$, $\delta_i : \mathbb{Q}_p^\times \to A^\times$ as i

Prop: A triangular (φ,Γ)-modules of rank 1 $/_{R_A}$ are isomorphic to $R_A(\delta)$, $\delta : \mathbb{Q}_p^\times \to A^\times$ for a δ unique.

Trianguline def. Functor

$$\mathcal{X}_{V,\mathcal{F}}(A) = \{ V_A \in \mathcal{X}_V(A) + \text{a triangulation of } D_{rig}(V_A) \text{ lifting the one of } D \text{ given by } \mathcal{F} \}.$$

(not a ~~sub~~functor) [à priori]

Theorem Assume that $\mathcal{F}$ is not critical, $\varphi_i \varphi_j^{-1} \notin \{1, p^{-1}\}$ if $i<j$.
Then $\mathcal{X}_{V,\mathcal{F}}$ is formally smooth of dim. $\frac{d(d+1)}{2}+1$ and there is an ex. seq:

$$0 \to \mathcal{X}_{V,\mathrm{cris}}(L[\varepsilon]) \to \mathcal{X}_{V,F}(L(\varepsilon)) \xrightarrow{\text{derivative of param.}} \mathrm{Hom}_{\mathrm{gr\,cont}}((\mathbb{Z}_p^\times)^d, L) \to 0.$$

Rk * Not obvious that the first map exists, $\mathcal{X}_{V,F}$ is a subfunctor of $\mathcal{X}_V$.
* First assumption relies on some comp. of Colmez of $H^i_{(\varphi,\Gamma)}(\delta)$. $i=0,1,2$.
* Exactness in the center means that
"A ~~noncritical~~ trianguline deform. of a non critical cryst. rep. is HT $\Leftrightarrow$ crystalline".

pf: show it is De Rham as in last prop. of lect. 1 $\Rightarrow$ pst by Berger $\xrightarrow{\text{easily}}$ result.
"infinitesimal version of Coleman's classicity result".
this uses non criticality as then the tri are $= k_i$ hence strictly incr.

Rk: * $\exists$ criterion to show that a given def. is trianguline
we will give it later and apply it to eigenvarieties.

(D) Global applications

- It is well known that any cont. rep. $\rho : G_{\mathbb{Q}} \to Gl_n(L\langle x \rangle)$ such that ρ_x is strongly geom., irreducible, is constant (Q-motives are countable)

- Infinitesimal version (general frame wk of BK conj, BSD conj)

$\bar\rho : G_{\mathbb{Q}} \to Gl_n(L)$ used geometric, ($\rho_{|G_{\mathbb{Q}_p}}$ crystalline).

CONJ then "any continuous "minimal geometric" ~~with good~~ $\rho : G_{\mathbb{Q}} \to Gl_n(L[\varepsilon])$ lifting $\bar\rho$ is trivial ($H^1_f(\mathbb{Q}, \mathrm{ad}\,\bar\rho) = 0$).

- Assume that $\bar\rho_{|G_{\mathbb{Q}_p}}$ crystalline etc and fix a refinement $\mathcal{F}$.

$\Rightarrow$ define $\mathcal{X}_{\bar\rho, \mathcal{F}}$ def. functor $\begin{cases} \text{minimal outside } p \\ \text{trianguline at } p. \end{cases}$

Then | if $\mathcal{F}$ non critical
| $\exists$ exact sequence

$$0 \to H^1_f(\mathbb{Q}, \mathrm{ad}\,\bar\rho) \to \mathcal{X}_{\bar\rho, \mathcal{F}}(L[\varepsilon]) \longrightarrow \underset{\text{cont gr.}}{\mathrm{Hom}}(\mathbb{Z}_p^{\times d}, L)$$

0 conjecturally $\Rightarrow$

should have dim $\leq d$ in principle.

interesting pb. predict the dim. of

- in some cases we can ($\bar\rho$ induced from a quadr. im. field + cond.)
- linked to dimension of eigenvarieties, last map is the map to weight space
- this shed some light on Coleman's theorem and the correct generalisation of the limit case.