April 13. 2006. Thursday 1:00 P.M. Kevin Buzzard

16-th lecture

smooth irred. $\mathbb{C}$-rep'n of $GL_2(\mathbb{Q}_p)$	$\longleftrightarrow$	F-semi-simple (ρ, N) 2-dim. WD rep'n
$\cup$		
1-dim	$\longleftrightarrow$	ρ s.t. a non-zero V exist. but $N = 0$

twist of St $\longleftrightarrow$ (ρ, N). $N \neq 0$ generic : infinite dimensional

$\boxed{1\text{-dim}}$ $\longleftrightarrow$ $(\rho, 0)$] non-generic

↑ are the 2 irred subquot of $I(\chi_1, \chi_2)$ when it's reducible

Local - Global compatibility

$$\begin{array}{ccc} f : \text{cusp form} & \longrightarrow & \pi_f \\ \downarrow & \circlearrowleft & \downarrow \text{LOC. LAN} \\ \rho_f & & (\rho, N) \\ & \searrow & \uparrow \text{semi-simplify} \\ & \rho_f|D_\ell & \xrightarrow{\text{Groth}} \text{WD rep} \end{array}$$

Rk: 1-dim case doesn't occur

At the end of last lecture

We were considering the mod ℓ picture:

How the non-generic case can occur in global setting.

Q) Is there a mod p local Langlands for $GL_2(\mathbb{Q}_\ell)$?

Observation: there is not such a correspondence, if we furthermore demand.

① irred mod p rep'n of $GL_2(\mathbb{Q}_\ell) \leftrightarrow$ certain Galois rep'n mod p

② Compatibility with reduction of classical local Langlands

Dumb. reason.

$\exists$ irred. char 0 rep'n of $GL_2(\mathbb{Q}_\ell) \longrightarrow Aut(V/\overline{\mathbb{Q}}_p)$ whose reductions are reducible

e.g. $I(\chi_1, \chi_2)$

$\chi_1, \chi_2 : \mathbb{Q}_\ell^\times \to \overline{\mathbb{Q}}_p^\times$

$\chi_i(\ell) = p$-adic unit $\qquad \chi_1/\chi_2 = |\cdot|^{\pm 1}$

$\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$

Jettison: ①

Allow possibility of reducible π's associated to ρ's

Local Langlands is becoming a "recipe" from ρ's to π's

Example: if ρ is cyclo $\oplus 1$, the mod p Galois rep'n $G_{\mathbb{Q}_\ell}$, then the associated π will have ≥ 2 J-H factors.

$\ell \not\equiv -1 \ (p)$ then 2

$\ell \equiv -1 \ (p)$ get 3

All but 1 are 1-dim'l.

and one is ∞-dim.

$I(\chi_1, \chi_2) \not\cong I(\chi_2, \chi_1)$

mod p characters of $\mathbb{Q}_\ell^\times$ if $\chi_1/\chi_2 = |\cdot|^{\pm 1}$

e.g. $\ell \not\equiv -1$ mod p, then one has a 1-dim sub. one has a 1-dim quotient.

Next step:

Local - Global

for overconvergent finite slope cuspidal eigen forms

$\ell \neq p$. p-adic MTTs. rep'n of $GL_2(\mathbb{Q}_\ell)$

Alex Paulin has constructed $\underline{\underline{\pi_{f,\ell}}}$ f: p-adic overconvergent eigen form wt $k \in \mathcal{W}$

$\frac{E_k(q^\ell)}{E_k(q)}$ overconv. wt 0. smooth adm. rep of $\underline{GL_2(\mathbb{Q}_\ell)}$.

ρ_f exists & we can ask how to relate

$$\rho_f|D_\ell \ \& \ \pi_{f,\ell}$$

- It seems that $\pi_{f,\ell}$ is not always irreducible.

& Similarly $\rho_f|D_\ell$ has <u>no reason</u> to be <u>generic</u>.

$\boxed{\ell + \text{level}}$

ρ_f could be unram @ ℓ,

& e-vals of $\rho_f(\text{Frob}_\ell)$ could be $\alpha, \beta \in \overline{\mathbb{Q}}_p^\times$, $\alpha/\beta = \ell$

$\pi_{f,\ell}$ = reducible unram P.S

$\text{C-Ind}_{GL_2(\mathbb{Z}_\ell)\mathbb{Q}_\ell^\times}^{GL_2(\mathbb{Q}_\ell)}$ Steinberg in the sub.

Again it looks like there's some kind of "correspondence"

$$\rho \longrightarrow \Pi$$

may not be irreducible.

<u>Remark</u>: The mod p story is connected to level-raising & lowering

① If F is a char 0 mod. form level N, $\ell \nmid N$.

& if $\overline{\rho_F}$ is the mod p rep'n

& $\overline{\rho_F}(\mathrm{Frob}_\ell)$ has e.vals α, β $\alpha/\beta = \ell$

then? G level $N\ell$. new at ℓ. (Ribet)

s.t $\overline{\rho_G} \equiv \overline{\rho_F}$?

② If G is a form of level $N\ell$.

new at ℓ

& if $\overline{\rho_G}$ is unramified @ ℓ then?

$\exists F$ level N s.t $\overline{\rho_G} = \overline{\rho_F}$?

yes in many cases (Mazur, Ribet)

In the p-adic Theory,

there are analogous questions

① If F is a family of Eigenforms of level N.

& $\ell \nmid Np$ & if $\rho_{F_k}(\mathrm{Frob}_\ell)$ has e.vals α, β

$\alpha/\beta = \ell$.

then? $\exists G$: family of forms of level $N\ell$.

generically new at ℓ.

but s.t $\rho_{G_k} = \rho_{F_k}$?

<u>$\ell = p$</u>

Terrifying things afoot.

Let $E/\mathbb{Q}$ be an elliptic curve with multiplicative red'n @ p.

Let f be the associated modular form

Q. Relate $\Pi_{f,p} \overset{?}{\longleftrightarrow} f$ (cond $= p$, char $= 1$) to $\rho_f|D_p$?

<u>Rmk</u>: $\rho_f|D_p$ has determinant the cyclo char, which is infinitely wildly ramified

$\pi_{f,p}$ will be a twist of St. & it'll ~~be~~ be unramified twist by a character of order 1 or 2

$\underline{\pi_{f,p} = St.}$ (split mult.) unram. quad. twist of St (non-split)

Split mult. case:

$\rho_f | D_p$ we can write it down

Tate Curve: $E(\overline{\mathbb{Q}_p}) \cong \overline{\mathbb{Q}_p}^\times / \langle q \rangle$. $q \in \mathbb{Q}_p^\times$. $|q| < 1$.

& q is determined by the fact that $\underline{j\text{-inv't of } E}$

is $j = q^{-1} + 744 + 196884\,q + \dots$ $|j| > 1$. $\mathbb{Q} \subseteq \mathbb{Q}_p$ $|q| = \frac{1}{|j|}$

$$\rho_f | D_p = \begin{pmatrix} \text{cyclo} & * \\ 0 & 1 \end{pmatrix}$$

q is a global object.

where $*$ is an ext'n of 1 by cyclo. determined by q.

Kummer Theory

$$q \in \varprojlim_n \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times n} \cong \mathbb{Z}_p^\times \times \mathbb{Z}_p$$

Lots of different $*$s can occur.

Split multi

Rep ~~by~~ side "St"

Galois by side: cbly. many possibility [Mazur–Tate–Teitelbaum, Greenberg–Stevens]

The hope that we can fix things up is dashed if we move to weight 4.

Set $p=5$. $N = 45$. $k = 4$: compute the new forms (trivial)

One example. $a_p^2 = p^{k-2}$

$f_1 = q - 5q^2 + 17q^4 + 5q^5 - 30q^7 + \dots$

$f_2 = q - 3q^2 + q^4 + 5q^5 + 20q^7 + \dots$

$\pi_{f_1,5} \cong \pi_{f_2,5}$ = unramified twist of Steinberg.

However,

$f_1 \underset{\text{mod } 5}{\equiv}$ form of wt 4 & level 9

$q - 8q^4 + 20q^7 + \cdots$

Hence $\overline{\rho_{f_1}}|_{D_5}$ is $\underset{\text{mod 5 rep'n}}{}$ irreducible. $\sim \mathrm{Ind}(\omega_2^2)$

OTOH .

$\rho_{f_2} \cong \text{cyclo} \otimes \rho_g$ $\quad g$ wt 6. Level 9. $g = q + 6q^2 + 4q^4 - 6 \cdot q^5 + \cdots$

& $\overline{\rho_{f_2}}|_{D_5}$ is reducible $\sim \begin{pmatrix} \omega^2 & * \\ 0 & \omega \end{pmatrix}$

$\overline{\rho_{f_1}|_{D_5}}$ & $\overline{\rho_{f_2}|_{D_5}}$ are completely different.

On the other hand.

T. Saito proved local-global compatibility at $p = \ell$ for classical modular forms.

[$\pi_{f,p}$ doesn't determin $\rho_f|_{D_p}$, but $\rho_f|_{D_p}$ does determine $\pi_{f,p}$]

Diversion

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$ be a conti. Galois rep'n

Fontaine defined a functor D_{st}

taking ρ to a "linear algebra object".

i.e. filtered (φ, N)-module.

"Filtered (φ, N)-module" $\qquad D_{st}(V) = \left(B_{st} \underset{\mathbb{Q}_p}{\otimes} V\right)^{G_{\mathbb{Q}_p}}$

① fin dim. $\overline{\mathbb{Q}}_p$-v. sp D

② A bijective $\overline{\mathbb{Q}}_p$-linear map $\varphi : D \to D$

③ A $\overline{\mathbb{Q}}_p$-linear endo. $N : D \to D$ s.t $N\varphi = p\varphi N$

④ A filtration $\mathrm{Fil}^i D$, $i \in \mathbb{Z}$

s.t $\mathrm{Fil}^i D \supseteq \mathrm{Fil}^{i+1}(D)$

$\bigcup_i \mathrm{Fil}^i D = D \qquad \bigcap_i \mathrm{Fil}^i D = D$

"Easy" to check that if $\rho : G_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V)$ is a cts rep'n,

then $\dim_{\mathbb{Q}_p}(D_{st}(\rho)) \leq \dim_{\mathbb{Q}_p}(V)$. If equality holds, V is said to be semi-stable

If ρ is semi-stable, then $D_{st}(\rho)$ is a filtered (φ, N)-module

& further $D_{st}(\rho)$ is weakly admissible

Weakly admissible

If D is a filtered (φ, N)-module

Define $t_H(D) = \sum_{i \in \mathbb{Z}} \dim\left(\frac{\mathrm{Fil}^i(D)}{\mathrm{Fil}^{i+1}(D)}\right) i$

& $t_N(D) = \sum_{\alpha \in \mathbb{Q}} (\dim D_\alpha) \cdot \alpha$

slope α generalized eigensp.

$t_N(D) = \sum_{\substack{\text{roots } x \\ \text{of char poly} \\ \text{of } \varphi}} v(x)$ $\quad v(p) = 1$.

D is Weakly admissible means

① $t_H(D) = t_N(D)$

② $t_H(D') \leq t_N(D')$ for all $(\varphi\text{-}N)$-stable subobject of D

i.e $D' \subseteq D$ st. ① $\varphi(D') \subseteq D'$

② $ND' \subseteq D'$

③ $\mathrm{Fil}^i D' = \mathrm{Fil}^i(D \cap D')$.

Fontaine checked

that if V was semi-stable

then $D_{st}(V)$ was weakly admissible.

& the functor $\begin{pmatrix} \text{Semi-stable} \\ \text{adm. rep'n} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{w.a. filtered} \\ \varphi, N\text{-modules} \end{pmatrix}$ was fully faithful

Much more recently, Fontaine & Colmez showed (1999) it was an equivalence of categories

Upshot. if V is semi-stable, we can recover V from $D_{st}(V)$

& we can "list" all the possibilities for $D_{st}(V)$

Rk. if f is a modular form of level N ($p \nmid N$) good reduction.
or Np & trivial char $\mathbb{Q}_p$.
multiplicative
then $\rho_f | D_p$ is <u>semi-stable</u>