Mar 7. 2006. Tuesday. Kevin Buzzard (E.V.) 17th lecture

Recall: I defined overconvergent p-adic modular functions & indicated that they were stable under Hecke operators.

$p=2$, $N=1$

$X_0(1)^{ord}$ = closed disc $\mathbb{Q}_2\langle P\rangle$

General weight

p: any prime

$N \geq 1$. $p \nmid N$

ftns here = p-adic modular ftns

p-adic Riemann Surface

$\mathbb{Q}_p^{nr}$-valued center

$X_1(N)_{/\mathbb{Q}_p} \supseteq X_1(N)^{ord} = X_1(N) \setminus$ (pts)

ell curves with s.s. reduction = fin. union of open discs

$\cup \qquad \cap$

$X_1(N)_{\geq p^{-r}}$ ← ftns here = r-overconvergent p-adic modular ftns

70's: Serre gave a definition of a p-adic modular form of wt k for $k \in (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$

Katz gave a definition of an overconvergent form. $(p>2)$ of wt $k \in \mathbb{Z}$

- from this viewpoint, an r-overconvergent form of wt k is a section of $\omega^{\otimes k}$ on $X_1(N)_{\geq p^{-r}}$

Where do Eisenstein Series fit into this picture

Washington: $\exists$ p-adic ζ-function

- same cong mod $p-1$
- drop an Euler factor.

- p-adic analytic extension of classical ζ-ftn

$\zeta(n)$, $n \in \mathbb{Q}$, $n < 0$
$\cap$
$\mathbb{Q}$ & "varies p-adically analytically with n as long as no fixed congruence class mod $p-1$ $\substack{2 \\ p=2}$, if $n \equiv 1$ mod $p-1$, Euler factor

Further on in Washington, $\exists$ much more beautiful explanation of p-adic ζ-fctn.

Define W = $\underline{\text{set}}$ of cts gp homs $\mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times$

W = " Weight space"

In fact W is naturally a p-adic Riemann Surface.

If $p > 2$, then $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$

& log $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ is a conti isomorphism of top. gps $1 + p\mathbb{Z}_p \xrightarrow{\cong} p\mathbb{Z}_p$

$p = 2$. . $\mathbb{Z}_2^\times \cong \{\pm 1\} \times (1 + 4\mathbb{Z}_2)$

$$\underset{(\mathbb{Z}/4\mathbb{Z})^\times}{\overset{\shortparallel}{}} \qquad \underset{4\mathbb{Z}_2}{\wr\!\wr \log}$$

Hence $\mathbb{Z}_p^\times \cong \underset{\substack{\shortparallel \\ G}}{\text{finite gp}} \times \mathbb{Z}_p$.

$$\therefore \quad W = \underset{\substack{\uparrow \\ \text{finite set}}}{\mathrm{Hom}(G, \mathbb{C}_p^\times)} \times \underset{\substack{\cup \\ 1}}{\mathrm{Hom}(\mathbb{Z}_p, \mathbb{C}_p^\times)}$$

A cts gp homo $\varphi : \mathbb{Z}_p \longrightarrow \mathbb{C}_p^\times$ is determined by $\varphi(1)$

$$\therefore \mathrm{Hom}(\mathbb{Z}_p, \mathbb{C}_p^\times) \subseteq \mathbb{C}_p$$
$$\varphi \longmapsto \varphi(1)$$

im $\not\ni p$

as if $\varphi(1) = p$, then $\varphi(-t) = p^{-t} \underset{t \in \mathbb{Z}}{\to} \underline{\underline{\infty}}$ but image is cpt of φ

$\therefore$ If $\varphi: \mathbb{Z}_p \to \mathbb{C}_p^\times$ is cts, then $|\varphi(1)| = 1$

$\therefore \overline{\varphi(1)} \in \overline{\mathbb{F}}_p^\times$ & it had better have p-power order $\therefore \overline{\varphi(1)} = 1$.

$|\varphi(1) - 1| < 1$.

Conversely, one checks that if $|t - 1| < 1$, then $\exists !\, \varphi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ cts gp homo s.t. $\varphi(1) = t$

Idea. $\varphi(p^n) = t^{p^n} \longrightarrow 1$

& after a while one can use exp & log to define

φ on $p^n\mathbb{Z}_p$ $\qquad \varphi(p^n s) = (t^{p^n})^s$

$$= \exp(s \times \log t^{p^n})$$

$\therefore\ \mathrm{Hom}(\mathbb{Z}_p, \mathbb{C}_p^\times)$ $\qquad$ If $n \gg 0$

7 $= \{t \in \mathbb{C}_p \mid |t-1| < 1\}$ $\qquad$ $\log(t^{p^n})$ is o.k.

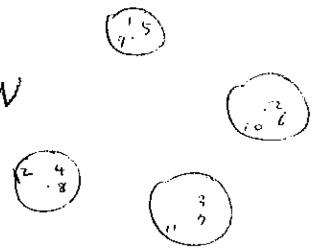
Hence wt sp W is naturally $\left\{\begin{matrix} p-1 & p>2 \\ 2 & p=2 \end{matrix}\right\}$ copies of an open disc

Examples of elements of W

$$\mathbb{Z} \hookrightarrow W \ni k$$

$$n \longmapsto \begin{pmatrix} x \longmapsto x^n \\ \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times \end{pmatrix}$$

$p=5$.



n close to m

$\Longrightarrow$ $n \equiv m \mod p-1$ (mod 2 if $p=2$)

& $n \equiv m \mod p^N$, N big

More generally if

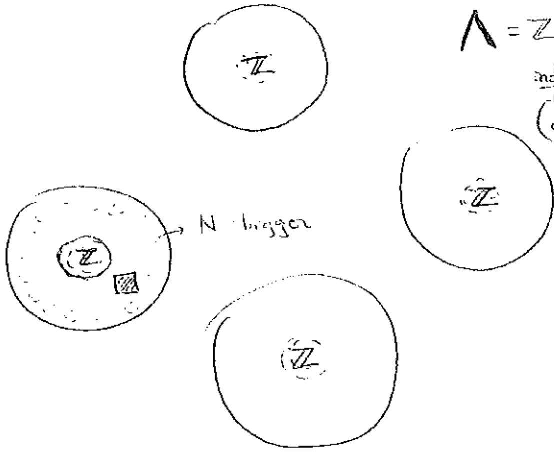
$\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \overset{\mathbb{C}}{\overset{\shortparallel}{\mathbb{C}_p^\times}}$ is a Dirichlet character & $k \in \mathbb{Z}$, then the

$n \geq 1$.

ftn $x \longmapsto x^k \cdot \chi(x)$ is an element of wt space.

$\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ $\qquad \varphi(1+p) = (1+p)^k \equiv L \mod p$

integral (bdd by 1 defined over $\mathbb{Q}_p$)



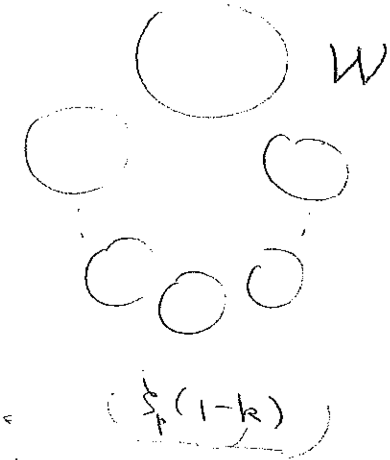
Rem: there are many more pts in W than those of the form (k, χ) as above.

Fact: the p-adic Zeta function is very naturally a function on W.

W has an even # of components as if $k \in W$ then $k(-1) = \pm 1$

& so $W = W^+ \amalg W^-$

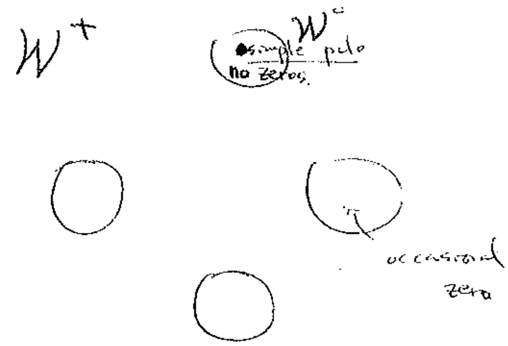
$\qquad\qquad k(-1)=1 \qquad k(-1)=-1$



__Fact:__ p-adic ζ-ftn is identically 0 on W^-

$W^+ \cong W^-$ $\qquad$ k : odd

[classical ζ-ftn vanishes at negative even integers



p-adic zeta function has a simple pole at the point $k=0$

i.e ftn $\mathbb{Z}_p^\wedge \longrightarrow \mathbb{C}_p^\times$

$x \longmapsto 1 \;\; \forall x$

Let W^0 denote the cpt of W that contains the zero character.

__Fact__ On W^0, p-adic zeta has no zeros

Indeed, its reciprocal is holomorphic & $\left|\frac{1}{\zeta_p(x)}\right| < 1$ for all $x \in W^0$.

On the other cpts of W^+, ζ_p has no poles but it could have zeroes.

It has zeros $\Leftrightarrow$ p is irregular.

__Recall.__ Classical Eisenstein make sense $k \geq 4$ even

$$E_k = \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n \qquad \text{MF. level 1}$$

__Note:__ the E_k's clearly do not "p-adically interpolates" to $k \in W$ because the ftn $\mathbb{Z}_{\geq 4} \longrightarrow \mathbb{C}_p^\times$, $k \longmapsto p^{k-1}$ doesn't extend to a p-adically cts ftn $W \longrightarrow \mathbb{C}_p^\times$.

To make nonconst coeffts. of E_k extend naturally over W, we drop an Euler factor.

So let's now ~~redefine~~ G_k

$$G_k = E_k(q) - p^{k-1} \cdot E_k(q^p)$$

$$(1+p^{k-1}) \cdot q^p - p^{k-1} \cdot q^p$$

Exer: $G_k = \dfrac{\zeta(1-k)}{2}(1-p^{k-1}) + \sum_{n\geq 1} \sigma^*_{k-1}(n) \cdot q^n$

$$\sigma^*_{k-1}(n) = \sum_{\substack{0<d\mid n \\ p\nmid d}} d^{k-1}$$

Easy Check:

For a fixed n the function $\mathbb{Z}_{\geq 4, \text{even}} \longrightarrow \mathbb{Q}_p$

$$k \longmapsto \sigma^*_{k-1}(n)$$

extends to a ftn on W^+

$d \in \mathbb{Z}$, $p \nmid d$, $d \in \mathbb{Z}_p^\times$

ftn $d \mapsto d^{k-1} = \dfrac{d^k}{d}$ extends to $W^+ \longrightarrow \mathbb{Q}_p$

$$\kappa \longmapsto \frac{\kappa(d)}{d}$$

Conclusion: Coeffs. of q^n, $n \geq 1$ of G_k extends to a ftn on W

By defn, p-adic ζ-ftn extends cst term.

So now we have a formal q-expansion

G_κ $\quad \kappa \in W^+$

$\kappa \neq$ zero char

$G_\kappa \in \mathbb{Q}_p[[q]]$.

Define $E_\kappa = G_\kappa \big/ \text{cst term}$

$$E_\kappa \in \mathbb{C}_p[[q]]$$
$$\parallel$$
$$1 + \cdots$$

E_κ is defined for $\kappa \in W^+$, κ not a zero of p-adic ζ-ftn

Rk: E_κ is well-defined on all of W^0 & $E_0 = 1$

If $k \in \mathbb{Z}$, $k \geq 4$, k: even
then E_k - i.e. new definition of E_k is a classical modular form of level $\Gamma_0(p)$

In fact if $k \in W^+$ is of the form $x \mapsto x^k \times \chi(x)$ as above & $k \in \mathbb{Z}$, $k \geq 1$ then the (purely formal) q-expansion E_k is the q-expansion of an Eisenstein series of wt k, level max $\{p, \text{cond}(\chi)\}$ & char χ.

If $\underset{W^+}{\overset{\in}{k}} \neq (k, \chi)$, then E_k is a purely formal q-expansion whose coeff vary analytically on W^+.

E_k were "known to exist" in 1970's

Coleman's insight:

General idea:
E_k should be an overconvergent MF of wt k.

DEFINE

an overconvergent MF of wt $k \in W^\circ$ to be a formal power series $F \in \mathbb{C}_p[[q]]$ st F/E_k is the q-expansion of an overconvergent modular ftn.

Rk. $E_0 = 1$
Our Def'n $\Rightarrow$ Overconv mod form wt 0 = overconv. mod. ftn

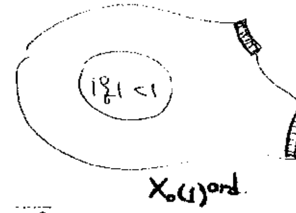
"$\times E_k$" is an isomorphism of vector space

$$\text{overconv. mod. ftns} \longrightarrow \text{overconv mod. forms wt } k$$

These spaces are sometimes "computable"
Several fundamental question instantly arise
i) If F has wt k & G has wt λ, does FG have wt $k\lambda$?
$\overset{\cap}{W^\circ}$

i.e. Is $\dfrac{E_k E_\chi}{E_{k\chi}}$ the q-exp of an overconvergent mod ftn?

<u>YES</u>



If F is classical, wt k, char χ, $k \geq 1$

is q-expansion of F overconvergent wt $\kappa = (k, \chi)$?

$\dfrac{E_s}{(E_{(1,\chi)})^s}$ is overconvergent

Yes. No so hard

F/E_k classical mod ftn level p^n <u>trivial char</u>

What about Hecke operators?

We know what Hecke operators do to q-expansions

eg if $F \in M_k(\Gamma_0(N))$

$$F = \sum_{n \geq 0} a_n q^n \qquad l \text{ prime } l \nmid N$$

$$T_l f = \sum a_{nl} q^n + \underset{\substack{\uparrow \\ \frac{\kappa(l)}{l}}}{l^{k-1}} \sum a_n q^{nl}$$

<u>Q</u> If F has wt κ level $\Gamma_0(N)$, $p \nmid N$

is $T_l F$ also wt κ?

<u>Yes</u> — One has to check that if E_κ is the Eisenstein series & l is <u>any</u> prime

then $\dfrac{E_\kappa(q)}{E_\kappa(q^l)}$ is indeed the q-expansion of an overconvergent modular <u>ftn</u>.

<u>Q</u>. How far does $\dfrac{E_\kappa(q)}{E_\kappa(q^l)}$ overconverge & how far does its reciprocal overconverge?