Feb 9. Thursday 1.00 PM — 2:30 PM. Kevin Buzzard 3rd lecture (E.V.)

~~Strand~~ 1

Eigenvarieties by examples.

- Serre "Endomorphismes Complètement continus des espaces de Banach p-adiques"

Let K be a field, complete wrt a non-trivial non-archimedean norm

e.g. $K = \mathbb{Q}_p$ finite ext'n

$\mathrm{Frac}(W(k))$ k any perfect field of char p

$K = \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$

$K = \mathrm{Frac}(F[\![t]\!])$ F : any field

A: Banach space over K. $\quad |\cdot|_K : K \to \mathbb{R}_{\geq 0}$

is a K-vector space V equipped with a norm ($|\cdot| : V \to \mathbb{R}_{\geq 0}$)

s.t. $|\lambda v|_V = |\lambda|_K \cdot |v|_V$

$|x+y|_V \leq \max\{|x|_V, |y|_V\}$

& V is complete wrt induced metric.

Key example of such a V.

Let I be any set.

Let V be the fns $f : I \to K$ $\quad$ ↪ equivalently $\lim_{i\to\infty} f(i) = 0$

s.t. $f(i) \to 0$ as $i \to \infty$

i.e. $\forall \varepsilon > 0, \varepsilon \in \mathbb{R}$, only fin. many $i \in I$ have $|f(i)| \geq \varepsilon$

V is a vector space in obvious way.

$|f| = \max_{i \in I} |f(i)|$

Exer) V is complete.

Examples of elements of V: If $i \in I$, then let e_i be the function $I \to K$ s.t. $e_i(j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $\qquad |e_i| = 1$.

The e_i's are a "basis".

For V – let's make this precise

$\boxed{\text{Def}}$ V: a Banach space $/K$

An orthonormal basis for V over K is a collection $\{e_i : i \in I\}$ of elements of V.

s.t. $|e_i| = 1 \quad \forall i \in I$

& for all $v \in V$, $\exists$ unique collection of $\lambda_i \in K$, $i \in I$,

s.t. $\lim_{i\to\infty} \lambda_i = 0$ & $\sum_{i\in I} \lambda_i e_i = v$

& furthermore $|v| = \max_i |\lambda_i| \in |K|_K$

$\boxed{\text{Rmk}}$: if V_I is the example above

$V_I = \{f : I \to K, f \to 0\}$

then the e_i are an ON basis for V_I,

Conversely, given V an O.N. basis $\{e_i\}_{i\in I}$ of V is an isom $V \cong V_I$

$\mathbb{Q}_p(\sqrt{p}) \to$ Not ON $\mathbb{Q}_p$-Banach sp.

$\mathbb{Q}_p$; $1+\sqrt{p}$

$|\sqrt{p}| \notin |\mathbb{Q}_p|$

If V & W are Banach spaces $/K$ then define $\mathcal{L}(V,W)$ = cts lin. maps $V \to W$

If $\varphi : V \to W$ is cts & linear, one checks that $\sup_{\substack{x \in V \\ x \neq 0}} \left(\frac{|\varphi(x)|}{|x|} \right) < \infty$ (sup in $\mathbb{R}_{\geq 0}$)

Let $|\varphi|$ denote the sup. $- " |\varphi|$

One checks that $\mathcal{L}(V,W)$ is now also a Banach space.

Now say V has ON basis $e_i : i \in I$

& W has ON basis $f_j : j \in J$

If $\varphi : V \to W$ is cts & linear then φ is determined by the a_{ji} where $\varphi(e_i) = \sum_{j \in J} a_{ji} f_j \quad a_{ji} \in K$.

Conversely When does a collection of $a_{ji} \in K$ give rise to a cts & linear map $V \to W$

Need

$|\varphi(x)| \leq |\varphi||x|$

① $\forall i$, $\lim_{j \to \infty} a_{ji} = 0$. "tends to 0 downwards"

② $\exists$ cnst C s.t. $|a_{ji}| \leq C \quad \forall i \forall j$

Conversely easy check shows that if (a_{ji}) satisfy ① & ② they come from a unique φ.

Examples

$I = J = \mathbb{N}$

"Matrix" is $\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \\ \vdots & \vdots & & \end{pmatrix}$

$\begin{pmatrix} \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix} & 0 \end{pmatrix}$ not o.k.

1_K

$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \\ & & & & \ddots \end{pmatrix}$ is o.k.

$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ & & 0 & & \end{pmatrix}$ is o.k.

$\varphi(e_i) = f_1 \quad \forall i$

$\varphi(\sum \lambda_i e_i) = (\sum \lambda_i) f_1$

$\sum \lambda_i e_i$ $\lambda_i \to \infty$ Complex theory
$\sum \lambda_i^2 < \infty$ L^2, ℓ^2.

"Compact endomorphism" ("Complètement continu")

V, W Banach spaces

$\mathcal{L}(V, W)$ is a Banach space

Def $\varphi : V \to W$ is finite rank, if $\mathrm{Im}\varphi$ is fin-dim

Let $\mathcal{F}(V, W)$ be a finite rank linear maps.

Closure of $\mathcal{F}(V, W)$ in $\mathcal{L}(V, W)$ is the compact linear maps

Rmk about norm on $\mathcal{L}(V, W)$

If V, W are ON able. basis e_i & f_j.

& $\varphi \in \mathcal{L}(V, W)$

then φ has a matrix & one can check $|\varphi| = \sup_{i,j} |a_{ji}|$

Matrix of a compact linear map

Fact: If $\varphi : V \to W$ is cpt, & V has basis e_i
W ——— f_j.

& (a_{ji}) is matrix of φ,

then ① $\sup_{i,j} |a_{ji}| < \infty$ (it's $|\varphi|$)

② $\lim_{j \to \infty} \sup_{i \in I} |a_{ji}| = 0$.

& Conversely, given any $a_{ji} \in K$ satisfying ① & ②

(a_{ji}) is matrix of a unique cpt operator φ.

② in words: cols $\to 0$ uniformly

② in picture: $\forall \varepsilon > 0$, $\varepsilon \in \mathbb{R}$

$\exists$ "horizontal cut-off" s.t $|a_{ji}| < \varepsilon$ beyond cut-off

$\forall \varepsilon$, $\exists$ line $\left(\dfrac{\qquad\qquad}{\text{ALL} < \varepsilon \text{ here}} \right)$

Non-example of cpt operator $I = J = \mathbb{N}$

$$\begin{pmatrix} 1 & 0 \\ 0 & \ddots \end{pmatrix}$$

Pf) Check that ① & ② are true for finite rank maps & stable under limits.
cpt (⇒) ① & ②

Converse: replace all entries beyond cut-off by 0.
cpt (⇐) ① & ②

Rmk. identity matrix $V \to V$ (infinite O.N. basis) in general has no trace

OTOH (on the other hand), a cpt matrix has a trace

Even better than trace

there is a good notion of "characteristic power series"

Don't do that $\det(X \cdot Id - \varphi)$ - this is a monic poly of degree = dim V

- instead do $\det(Id - X\varphi)$ — in f'd case, its usual char poly. written backwards.

$= 1 - (\mathrm{tr}\,\varphi)X + \cdots$

Let's construct $\det(I\cdot d - X\cdot\varphi)$ for φ cpt : $V \longrightarrow V$

Defn: If $S \subseteq I$ is a finite subset V O.N. basis e_i, $i \in I$

& $\sigma: S \to S$ is a bijection.

Set $n_{S,\sigma} = \prod_{i \in S} a_{i,\sigma i}$

$c_S = \sum_{\sigma: S \to S} \mathrm{sgn}(\sigma)\, n_{S,\sigma}$

$m \in \mathbb{Z}_{\geq 0}$ $\quad C_m = (-1)^m \sum c_S$

Case: $m=0$. $C_0 = 1$

$m=1$. $C_1 = -\sum a_{ii}$ (converges)

$|a_{ji}| \le B$

$< \frac{\varepsilon}{B^{m-1}}$

Exercise: sum defining C_m converges.

Defin of $\det(\mathrm{Id} - X\cdot\varphi)$ is $\sum_{m\ge 0} c_m X^m$.

$\Lambda^m V \to \Lambda^m W$ $\qquad = 1 - (\mathrm{tr}\,\varphi)X + \cdots$

Example: diagonal matrices

$I = J = \mathbb{N}$.

$$\varphi = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & d_3 & \\ 0 & & & \ddots \end{pmatrix}$$

$\varphi(e_i) = d_i e_i \quad d_i \in K$.

φ cts $\iff |d_i|$ bdd.

φ cpt $\iff d_i \to 0$ as $i \to \infty$

& in this case. $\det(\mathrm{Id} - X\cdot\varphi)$

$= \prod_{i\ge 1} (1 - d_i X)$

– think of this as converging in $\mathcal{O}_K[[X]] \otimes K$

$\left(\sum a_n X^n \text{. } a_n \text{ bdd. with norm } \sup_n |a_n| \right)$

Rmk.

If R is a comm. ring.

L is free R-module.

& $\varphi : L \to L$ is R-linear.

say φ is fin. rank, if $\mathrm{Im}\,\varphi \subseteq$ f.g. sub R-module of L

In this case. $\mathrm{Im}\,\varphi \subseteq M$. M: f.g. ~~free~~ R-module s.t $\exists N$ $L = M \oplus N$

$$\begin{pmatrix} (*) & 0 \\ 0 & \end{pmatrix}$$

Consider $\varphi|_M : M \to M$

$\det(\mathrm{Id} - X\cdot\varphi)$ makes sense

Easy checks: indep. of M.

If valn is non-trivial

& $\varphi : V \to V$ is cpt.

Scale φ s.t $|\varphi| \le 1$. Then $a_{ij} \in \mathcal{O}_K = \{x \in K : |x| \le 1\}$

& If $I \subseteq \mathcal{O}_K$ is principal & non-zero. φ mod I is fin. rk.

$\det(\mathrm{Id} - X(\varphi \bmod J)) \in (\mathcal{O}_K/J)[[X]]$ char power ser.

lim as $J \to \infty$ recover CPS.

One thing that a char power series is good for

$\varphi : V \to V$ cpt. V an O.N. basis

$$\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \pi & 0 & \\ & & \pi^2 & \ddots \end{pmatrix}$$

$F(x) = \det(\mathrm{Id} - X\varphi) =: \mathrm{CPS}(\varphi) \in K[[X]]$

Say $a \in K$ is a zero of $F(X)$ of order h.

If $\mathrm{char}(K) = 0$, this means $F^{(n)}(a) = 0$, $\forall n < h$.

but $F^{(h)}(a) \neq 0$.

$$F^{(n)}(x) = \frac{d^n F}{dx^n}. \qquad F = 1 + \cdot$$

Fact: V then decomposes as a direct sum $V = N \oplus F$ of closed φ-invariant subspaces with $\dim N = h$

& s.t. $1 - a\varphi$ is nilpotent on N,

& invertible on F (cont. inverse)

Rmk about convergence.

$\varphi : V \to V$ cpt.

$F(X) = \mathrm{CPS}(\varphi)$

Then $\forall x \in K$

$F(x)$ converges!

One can give an explicit proof by looking @ defn of C_m. I gave above

$|C_m| \to 0$ quicker than any poly in m.

Finish with 2 examples of cpt endomorphisms of V.

V: ON basis e_i, $i \in \mathbb{N}$

$$\varphi_1 = 0 = \begin{pmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$$\varphi_2 = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & \pi & & 0 & \\ & & 0 & \pi^2 & & \\ & & & 0 & \ddots & \\ 0 & & & & \ddots & \pi^n \end{pmatrix} \qquad \pi \in K, \ |\pi| < 1$$

$\mathrm{CPS}(\varphi_1) = \mathrm{CPS}(\varphi_2) = 1$

$\mathrm{Im}\,\varphi_1 = 0$

$\mathrm{Im}\,\varphi_2$ is non-closed dense subspace of V.

example

$$\varphi = \begin{pmatrix} 1 & & & \\ & \pi & & \\ & & \pi^2 & \\ & & & \pi^3 \ddots \end{pmatrix}$$

$\mathrm{CPS}(\varphi) = \prod_{n=1}^{\infty} (1 - \pi^n X)$

$a = \pi^{-n}$ is a zero of order 1 $\forall n \geq 0$

$N = K \cdot e_{n+1}$, F = basis e_i $i \neq n+1$

$\varphi = \pi^n$ on N.

cpt ✓

inj ✓

dense image ✓

CPS = 1 ✓

$\varphi(e_i) = \pi^{?} e_{\sigma i}$

$\sigma : \mathbb{N} \to \mathbb{N}$ nontrivial ... finite

&. matrix for $1-a\phi$ on F is diagonal with non-zero entries almost all of which invariant 1.