Apr. 12th Galois Representations & (φ,Γ)-modules (2)

by Berger

- $\{\mathbb{F}_p\text{-reps of } G_{\mathbb{Q}_p}\} \longleftrightarrow \{(\varphi,\Gamma)\text{-modules over } \mathbb{F}_p((X))\}$

⑤ The operator ψ and $D^{\#}$

$\varphi(\mathbb{F}_p((X))) = \mathbb{F}_p((X^p))$

$\mathbb{F}_p((X)) = \bigoplus_{i=0}^{p-1} (1+X)^i \, \mathbb{F}_p((X^p))$

If D is a φ-module over $\mathbb{F}_p((X))$, $D = \bigoplus_{i=0}^{p-1} (1+X)^i \varphi(D)$

If $y \in D$, $\exists!\ y_0, y_1, \ldots, y_{p-1}$ s.t. $y = \varphi(y_0) + (1+X)\varphi(y_1) + \cdots + (1+X)^{p-1}\varphi(y_{p-1})$

Define ψ on D by $\psi y = y_0$

Example: if $D = \mathbb{F}_p((X))$

$$\psi\Big(\sum_{i\in\mathbb{Z}} a_i (-X)^i\Big) = \sum_{i\in\mathbb{Z}} a_i (-X)^{\lfloor \frac{i}{p} \rfloor}$$

<u>Prop.</u> (1) If $\lambda \in \mathbb{F}_p((X))$, $y \in D$, then $\psi(\varphi(\lambda)\, y) = \lambda \psi(y)$

(2) If $\gamma \in \Gamma$, then $\psi \circ \gamma = \gamma \circ \psi$

One can also define ψ in char 0, and then (1) & (2) hold and

$$\psi(f)\big((1+X)^p - 1\big) = \frac{1}{p} \sum_{\eta^p = 1} f\big((1+X)\eta - 1\big)$$

- ψ decreases the denominators

<u>Prop.</u> If D is a φ-module over $\mathbb{F}_p((X))$, then $\exists!$ $\mathbb{F}_p[[X]]$-module $D^{\#} \subseteq D$ such that (1) $D^{\#}$ is a lattice in D

(2) If $y \in D$, $\exists\, m(y)$ s.t. $\psi^{m(y)}(y) \in D^{\#}$

(3) $\psi : D^{\#} \to D^{\#}$ is surjective

For example, let $D = \mathbb{F}_p((X))$.

If $j \geq 1$, $\psi(X^{-(j+1)}\mathbb{F}_p[\![X]\!]) \subset X^{-j}\mathbb{F}_p[\![X]\!]$

& $\psi(X^{j}\mathbb{F}_p[\![X]\!]) \supseteq X^{j-1}\mathbb{F}_p[\![X]\!]$

$\Rightarrow (\mathbb{F}_p[\![X]\!])^{\#} = X^{-1}\mathbb{F}_p[\![X]\!]$

- In char 0, there's an analogous result with
  - (1') $\forall n \geq 1$ $D^{\#}/p^n$ is a lattice in D/p^n
  - (2') $\forall n \geq 1$ $\forall y \in D$, $\exists m(y,n)$ $\psi^{m(y,n)}(y) \in p^n D + D^{\#}$
  - (3') unchanged

⑥ Representations of $B = \begin{pmatrix} * & * \\ & * \end{pmatrix} \subset G = GL_2(\mathbb{Q}_p)$

- If D is a (φ,Γ)-module over $\mathbb{F}_p((X))$
  - $\left(\varprojlim_{\psi} D\right)^{b} = \left\{ (y_0, y_1, \ldots), \begin{array}{l} \text{bounded for the } X\text{-adic topology} \\ \psi(y_{i+1}) = y_i \quad \forall i \geq 0 \end{array} \right\}$

$\quad\quad\quad\quad \cup$

If y, $\exists n \geq 0$ s.t. $y_i \in X^{-n}D^{\#}$ for $\forall i \geq 0$

- $\exists m > 0$ s.t. $\psi^m(X^{-n}D^{\#}) \subset D^{\#}$

$y_i = \psi^m(y_{i+m}) \Rightarrow y_i \in D^{\#}$ $\forall i \geq 0$.

$\Rightarrow \left(\varprojlim_{\psi} D\right)^{b} \longleftarrow \varprojlim_{\psi} D^{\#}$

- Choose a character η of $\mathbb{Q}_p^{\times}$ ($\eta : \mathbb{Q}_p^{\times} \to \mathbb{F}_p^{\times}$) and define an action of B on $\varprojlim_{\psi} D^{\#}$

If $y \in \varprojlim_{\psi} D^{\#}$

$\begin{pmatrix} a & \\ & a \end{pmatrix} y = \eta^{-1}(a)\, y$, $\left[\begin{pmatrix} 1 & \\ & p^{-j} \end{pmatrix} y\right]_i = \psi^{j}(y_i) = y_{i-j}$

$\left[\begin{pmatrix} 1 & \\ & a \end{pmatrix} y\right]_i = \gamma(y_i)$ where $\chi(\gamma) = a^{-1} \in \mathbb{Z}_p^{\times}$

$$\left[\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} y\right]_i = (1+X)^{p^i z} y_i \quad , z \in \mathbb{Z}_p$$

This makes $\varprojlim_{\psi} D^{\#}$ into a compact rep of B

<u>Theorem</u> (1) If V is a 2-dim^l mod p rep of $G_{\mathbb{Q}_p}$, if $\pi(V)$ is the rep of G associated to V by the mod p Langlands correspondence

$\downarrow$ "Colmez Isom" (Colmez, B-B)

then $\Pi(V)^{*} \underset{B}{\sim} \left(\varprojlim_{\psi} D^{\#}(V)\right)^{*}$ ← their semisimplifications are equal

(2) If V is a 2-dim $\mathbb{Q}_p$ trianguline rep of $G_{\mathbb{Q}_p}$, and if $B(V)$ is the rep of G assoc. to V by the p-adic LLC,

then, $B(V) \underset{B}{\sim} \left(\varprojlim_{\psi} D^{\#}(V)\right)^{b*}$

⑦ Characters & parabolic inductions

ω = mod p cyclotomic char.

Γ_λ = unramified char. $\underset{(\alpha \mapsto \alpha^p)}{\mathrm{Frob}_p} \longmapsto \lambda^{-1}$

$V = \omega^r \Gamma_\lambda$

$D(V) = \mathbb{F}_p((X))\, e$. $\quad \varphi e = \lambda e$

$\qquad \gamma e = \omega^r(\gamma) e$

$D^{\#}(V) = X^{-1}\mathbb{F}_p[\![X]\!]\, e$

$\varprojlim_{\psi} X^{-1}\mathbb{F}_p[\![X]\!]\, e \supseteq \varprojlim_{\psi} \mathbb{F}_p[\![X]\!]\, e$ △ - irred. sub representation

Let $\Omega_\eta(V) = \left(\varprojlim_{\psi} \mathbb{F}_p[\![X]\!]\, e\right)^{*}$

Then, we have

$$0 \to \eta\omega^{1-r}\mu_{\lambda^{-1}} \otimes \omega^{r-1}\mu_\lambda \to \left(\varprojlim_\psi D^\#(V)\right)^* \to \Omega_\eta(V) \to 0 \qquad (1)$$

$$\left(\chi_1 \otimes \chi_2 : B \longrightarrow \mathbb{F}_p^\times \atop \begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d) \right)$$

- What is $\Omega_\eta(V)$?

Consider $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$

$$0 \to \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)_0 \to \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2) \to \chi_1 \otimes \chi_2 \to 0$$

$$\sigma \longmapsto \sigma(\mathrm{Id})$$

<u>Theorem</u> If $V = \omega^r\mu_\lambda$, $\Omega_\eta(V) \approx \mathrm{Ind}_B^G(\omega^r\mu_\lambda \otimes \eta\omega^{-r}\mu_{\lambda^{-1}})_0$

Proof: <u>Step A</u>. $LC_0(\mathbb{Q}_p, \mathbb{F}_p) = \{f : \mathbb{Q}_p \to \mathbb{F}_p,$ loc. const, compactly supp$\}$

$\sigma \in \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)_0 \rightsquigarrow f_\sigma(z) = \sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}\right)$

Since $G = B \amalg B\begin{pmatrix} 0 & 1 \\ -1 & * \end{pmatrix}$

$\sigma \mapsto f_\sigma$ is a bijection.

We have $f_{\begin{pmatrix} a & b \\ & d \end{pmatrix}\sigma}(z) = \chi_1(d)\chi_2(a) f_\sigma\left(\frac{dz-b}{a}\right) \qquad (2)$

<u>Step B</u>: The Amice transform

$\mathcal{D}$ measure on $\mathbb{Z}_p$, i.e. $\mu : LC(\mathbb{Z}_p, \mathbb{F}_p) \longrightarrow \mathbb{F}_p$

$\mathcal{A}(\mu) \in \mathbb{F}_p[\![X]\!]$

$$\mathcal{A}(\mu) = \sum_{n=0}^{\infty} \mu\left(z \mapsto \binom{z}{n}\right) \cdot X^n = \int_{\mathbb{Z}_p} (1+X)^z \, d\mu.$$

- $\{z \mapsto \binom{z}{n}\}_{n \geq 0}$ is a basis of $LC(\mathbb{Z}_p, \mathbb{F}_p)$

$\Rightarrow \mu \mapsto \mathcal{A}(\mu)$ is bijection

Use this to let Γ, ψ, ... act on the space of measures

$$\int_{\mathbb{Z}_p} f(z)\, d(\gamma\mu) = \int_{\mathbb{Z}_p} f(\chi(\gamma) z)\, d\mu \qquad (3)$$

$$\int_{\mathbb{Z}_p} f(z)\, d(\psi\mu) = \int_{p\mathbb{Z}_p} f(p^{-1} z)\, d\mu \qquad (4)$$

<u>Step C</u>. If $y \in \varprojlim_{\psi} \mathbb{F}_p[\![X]\!]$, $y = (f_i(X)\cdot e)_{i \geq 0}$

$$\psi(\lambda^{-i} f_i(X)) = \lambda^{-(i-1)} f_{i-1}(X)$$

Define a measure $\mu_{y,i}$ on $\mathbb{Z}_p$ by

$$\mathcal{A}(\mu_{y,i}) = \lambda^{-i} f_i(x)$$

$\rightsquigarrow$ Can glue them to define a measure μ_y on $\mathbb{Q}_p$

if $f \in LC_0(\mathbb{Q}_p, \mathbb{F}_p)$ has support in $p^{-i}\mathbb{Z}_p$,

then $\int_{\mathbb{Q}_p} f\, d\mu_y = \int_{\mathbb{Z}_p} f(p^{-i} z)\, d\mu_{y,i}$ does not depend on i by (4)

$\Rightarrow (\varprojlim_{\psi} \mathbb{F}_p[\![X]\!] e)^* \simeq (LC_0(\mathbb{Q}_p, \mathbb{F}_p))^*$

<u>Step D</u>. Finally $g \in B$, $\sigma \in \mathrm{Ind}_B^G(\mu_\lambda \omega^r \otimes \eta \mu_{\lambda^{-1}} \omega^{-r})_0$

$y \in (\varprojlim_{\psi} \mathbb{F}_p[\![X]\!] e)$

$$\Rightarrow \int_{\mathbb{Q}_p} f_{g\sigma}\, d\mu_{gy} = \int_{\mathbb{Q}_p} f_\sigma\, d\mu_y \qquad \square$$

⑧ The mod p Langlands correspondence

$V = \mu_\lambda \omega^{r+1} \oplus \mu_{\lambda^{-1}}$ (by twisting)

central character $\eta = \omega^r$

(1)

$$\left[\varprojlim_{\psi} D^{\#}(\mu_\lambda \omega^{r+1} \oplus \mu_{\lambda^{-1}})\right]^* \sim \mathrm{Ind}_B^G(\mu_\lambda \omega^{r+1} \otimes \mu_{\lambda^{-1}} \omega^{-1})_0 \oplus (\mu_{\lambda^{-1}} \otimes \mu_\lambda \omega^r)$$

$$\oplus\, \mathrm{Ind}_B^G(\mu_{\lambda^{-1}} \otimes \mu_\lambda \omega^{\lambda})_0 \oplus (\mu_\lambda \omega^{r+1} \otimes \mu_{\lambda^{-1}} \omega^{-}$$