Arithmeticity (or not) of Monodromy

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In 1974 Griffiths and Schmid [1] asked whether monodromy groups of families of varieties acting on cohomology are arithmetic or not. The problem remains largely open, even for well known explicit examples. One case is that of the the families of Calabi-Yau three-folds, which have received much attention starting with the paper of Candelas-Parks *et al.* Of the well known 14 such families, 7 are known to be not arithmetic (Brav and Thomas [3]), while 3 are arithmetic (Singh -Venkataramana [5]) and 4 remain undecided. Besides deciding the remaining 4 cases the question is:

What is the geometric significance of being arithmetic?

Some further comments on this problem taken from Peter Sarnak's e-mail correspondence. Sarnak writes:

1. My interest in whether such groups are arithmetic or "thin" (the image of the monodromy group H is contained in the integer points $G(\mathbf{Z})$ of its Zariski closure. I call the group H **thin** if it is not finite index in the latter group, and arithmetic otherwise). All this is explained in my "Notes on Thin Matrix groups." While the main "expansion theorem" allows one to proceed in many cases without knowing whether the group is thin or not (and only knowing the Zariski closure—just as with most arithmetic geometric applications), in the diophantine orbit world one does want to know more. E.g if the group happens to be arithmetic then one can often use automorphic forms, Galois cohomology and other methods to resolve a problem. These are much more powerful than the substitute theory for "thin" groups.

2. A second reason to be interested in this is curiosity. That is, can one really compute a monodromy group? The first question after the Zariski closure is whether it is thin or not. The general question of whether the group is thin or not has no decision procedure. The situation is very similar to Hilbert's 10-th problem for decision procedures for diophantine problems. That is the local obstructions are the finite quotients of the (say, monodromy) group H. These can be identical to a determinable congruence subgroup $K \subset G(\mathbf{Z})$ even when H is of infinite index in $G(\mathbf{Z})$. So in this case one passes all local tests; so now, how to tell if H is this unique congruence subgroup K or is thin? If it is K, one might certify it is so by exhibiting the generators of K as products of the generators of H (which is, in fact, then manner in which H is given). The analogue of this in Hilbert 11 is that one gives an integral point on the variety demonstrating it has a point. The problematic step is giving

a certificate that H is thin. One method (which I view as the analogue of the method of descent) is to show that the generators of H play ping pong on some set. This means that there is a way of seeing expressions in the generators are getting more complex and hence of understanding the structure of H with its generators combinatorially. This can often be combined with cohomological methods to give a certificate that a group is thin. The trouble with this is it hard to show that the generators do, in fact, play ping pong. In some examples this can be done and this is what Brav and Thomas did for the Dwork $n = 4$ case. My talk will be about a certificate for being thin, which is a bit like the Brauer-Manin obstruction. It applies to the hypergeometric monodromy groups $n\_F_(n-1)$ when their Zariski closure is orthogonal of signature $(n-1, 1)$ and it is quite robust.

3. I don't have any good ideas about the geometric significance of being thin or not (and I would love to hear some ideas). However my feeling is that being thin is exotic enough that in some examples it is the reason it carries precious information (somehow if the the group were arithmetic the information gleaned would have been extracted by other means). To back this up note that the Dwork Family used by Taylor et al (with n equal to 4 and higher) is I expect thin. For $n = 4$ this is proved by Brav-Thomas in [3]. Also the Candelas case—which set off mirror symmetry story—is thin. Kontsevich has some ideas coming from dynamics connected to variation of Hodge structures that might explain the significance of thin. See the report on his recent lecture [4].

References

[1] P. Griffiths and W. Schmid in "Discrete subroups of Lie groups and applications to moduli," Oxford Press (1975), 31-129.

[2] C. Doran and J. Morgan, "Mirror symmetry and integral variations of Hodge structure under one parameter families of Calabi-Yau threefolds," arXiv/math:0505272 (2005).

[3] C. Brav and H. Thomas "Classical and derived monodromy of quintic threefolds," arXiv:1210.0523 (2012).

[4] M. Kontsevich, http://matheuscmss.wordpress.com/2013/01/09/maxim-kontsevichs-talk-on-lyapunov-exponents-for-variations-of-hodge-structures/

[5] S. Singh and T. Venkatamarana "Arithmeticity of certain symplectic hypergeometric groups," arXiv:1208.6460 (2012).