Whittaker models and Fourier coefficients of automorphic forms

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- Set $\mathfrak{g} = Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$. We will use the notation $U(\mathfrak{g})$ and $Z(\mathfrak{g})$ respectively for the universal enveloping algebra of $\mathfrak{g}$ (thought of as left invariant vector fields) and the center of $U(\mathfrak{g})$.

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- Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ has finite volume. Then a Γ-automorphic form on G is a function $f \in C^\infty(\Gamma\backslash G)$ satisfying the following three conditions:

1. Set $R_g f(x) = f(xg)$ then $\dim Span_{\mathbb{C}} R_K f < \infty$.
2. $\dim Z(\mathfrak{g}) f < \infty$.
3. There exits r and for each $x \in U(\mathfrak{g})$ there is a constant C_x such that $|xf(g)| \leq C_x \|g\|^r$.

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- In the case of $SO(n_1, 1) \times \cdots \times SO(n_k, 1)$ with $k > 1$ the conjecture is true.

Example

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- Then $Z(\mathfrak{g})$ is generated by one element the Casimir operator, C.
- G/K is the upper half plane, $\mathcal{H}$, with G acting by linear fractional transformations.

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- and if $\Gamma \cap N = \left\{ \begin{bmatrix} 1 & nh \\ 0 & 1 \end{bmatrix} | n \in \mathbb{Z} \right\}$. h is called the height of the cusp at infinity. Then $\phi(z+h) = \phi(z)$ so if we set

$$\tau = e^{\frac{2\pi i z}{h}}$$

then we can write

$$\phi = \sum a_n \tau^n.$$

- If we set

$$f(g) = (ci+d)^{-l}\phi\left(\frac{ai+b}{ci+d}\right)$$

then $f(\gamma g k(\theta)) = f(g)e^{il\theta}.\gamma \in \Gamma$ and $k(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ so $f \in C^{\infty}(\Gamma\backslash G)$ and satisfies 1.

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- Finally if $a_n = 0$ for $n < 0$ then f satisfies 3.

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- If f is a Γ–automorphic form then we set $V_f = Span_{\mathbb{C}}(U(\mathfrak{g})R_K f)$. Then V_f is a $(\mathfrak{g}, K)$–module that is admissible (a consequence of basic theorem of Harish-Chandra).

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- In the example above the corresponding representation for $l > 1$ is a holomorphic discrete series representation.

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- If V is a Fréchet space and the map $g \longmapsto \pi(g)v$ is C^∞ then we call (π, V) a smooth Fréchet representation. Differentiation yields a representation of $U(\mathfrak{g})$. We set

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- V is said to be admissible (resp. finitely generated) if V_K is admissible (resp. finitely generated).
- We say that V is of moderate growth if for each continuous semi-norm λ on V there exists a continuous semi-norm p_λ on V and a constant r such that $|\lambda(\pi(g)v)| \leq \|g\|^r p_\lambda(v)$.

- For example, define $C_{u\,\mathrm{mod}}^{\infty}(G)$ to be the space of all $f \in C^{\infty}(G)$ such that $|xf(g)| \leq C_x \|g\|^{r_f}$ for $x \in U(\mathfrak{g})$, and C_x depending on f and x. We set $C_r^{\infty}(G)$ equal to the space of all f such that we can take $r_f = r$. We define the seminorms

$$p_{r,x}(f) = \sup \frac{|xf(g)|}{\|g\|^r}.$$

Defining a Fréchet space topology on $C_r^{\infty}(G)$. We endow $C_{u\,\mathrm{mod}}^{\infty}(G)$ with the direct limit topology (depending on the r that occurs).

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- Let f be a Γ–automorphic form. In the induced topology the closure $\overline{V_f}$ defines a smooth, admissible,finitely generated Fréchet representation of moderate growth.

- Let $\mathcal{C}(\mathfrak{g}, K)$ be the category of admissible, finitely generated $(\mathfrak{g}, K)$–modules and $\mathcal{F}_{\mathrm{mod}}(G)$ be the category of smooth, admissible, finitely generated, Fréchet representations of moderate growth.

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- If the topological space of a representation of G is a Hilbert space then the representation is called a Hilbert representation. If (π, H) is a Hilbert representation then $v \in H$ is called a C^∞ vector if the map G to H given by $g \mapsto \pi(g)v$ is C^∞. Let H^∞ denote the space of C^∞vectors.

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- Then as above there is an action of $U(\mathfrak{g})$ on H^∞. We endow H^∞ with the topology defined by the seminorms $p_x(v) = \|\pi(x)v\|$ for $x \in U(\mathfrak{g})$. Then H^∞ is a smooth, Fréchet representation of moderate growth. If H^∞ is admissible or finitely generated then we say that (π, H) is admissible or finitely generated.

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- In the $SL(2,\mathbb{R})$ example, if $l \geq 1$ and if $f \in L^2(\Gamma\backslash G)$ then one can see that in fact $a_0 = 0$ and this implies that f is a cusp form. We will soon explain the other a_k in terms of Whittaker models.

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- We use the above material to give C^∞ definition of a Γ–automorphic form. Let (π, V) be an admissible, finitely generated smooth Fréchet representation of moderate growth. Let V' be the continuous dual space of V and let

$$\lambda \in \left(V'\right)^\Gamma = \{\mu \in V' | \mu \circ \pi(\gamma) = \mu, \gamma \in \Gamma\}.$$

Then $f(g) = \lambda(\pi(g)v)$ defines an automorphic form if $v \in V_K$. Furthermore, every automorphic form is obtained in this fashion.

Whittaker models

- We first consider the case of $SL(2,\mathbb{R})$ as above. Let f be an automorphic form that corresponds to a holomorphic automorphic form on the upper half plane. Then $f(\gamma g) = f(g)$ for $\gamma \in \Gamma$ hence for $\gamma \in N \cap \Gamma$. Thus if

$$n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

the function (in x)

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is periodic of period h. We can define

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- Then $f_k(n(x)g) = e^{\frac{2\pi ikx}{h}} f(g).f_k \neq 0$ only if $k \geq 0$ and the function f_k corresponds to $a_k \tau^k$.

- Let $P = MN$ be a parabolic subgroup of G (M a Levi factor and N the unipotent radical). E.g. $G = SL(n, \mathbb{R})$ and M the group of all (fixed) block diagonal matrices and N the group of all matrices that are identity on the block of the block diagonal and 0 below the diagonal.

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- Let $\chi : N \to S^1$ be a unitary character and let $d\chi : Lie(N) \to i\mathbb{R}$ be its differential. If $V \in \mathcal{C}(\mathfrak{g}, K)$ then a χ–Whittaker vector is an element of $\eta \in V^*$ such that $\eta(Xv) = d\chi(X)\eta(v)$ for $X \in Lie(N)$, $v \in N$. Let $W_\chi(V)$ be the space of all such η.

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- Let $W_\chi(Cl(V))$ be the elements $\lambda \in W_\chi(V)$ that extend to continuous functionals on $Cl(V)$.
- Let G be quasi-split, V be irreducible, P be minimal (i.e. a Borel subgroup). If χ is generic (i.e. the stabilizer of χ in M is trivial) then $W_\chi(Cl(V))$ is at most one dimensional (for $SL(n, \mathbb{R})$ this is due to Jacquet and Shalika).

- In the case of $SL(2,\mathbb{R})$ if (π, H) is a holomorphic discrete series representation and of P is the parabolic subgroup corresponding to the upper triangular matrices then if Γ is a subgroup of $SL(2,\mathbb{Z})$ with h the width of its cusp at infinity. Then if $\chi(n(x)) = e^{\frac{2\pi ikx}{h}}$ and $k > 0$ then χ is generic and $W_\chi(H^\infty)$ is one dimensional so equal to $\mathbb{C}\eta$.

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- If $\lambda \in ((H^\infty)')^\Gamma$ then if $v \in H_K^\infty$ corresponds to the K–type l (the minimal K–type) we have

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Up a fixed scalar a_k is given by $c_k(\Gamma)$.

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- There is a similar multiplicity one theorem for Whittaker modules holomorphic discrete series, (π,H), corresponding to Hermitian symmetric spaces of tube type.

- For the corresponding groups there is a parabolic subgroup of G with abelian unipotent radical, N. One shows that for generic characters χ of N the representation of the stabilizer of χ in M on $W_\chi(H^\infty)$ is multiplicity free and if the minimal K–type is one dimensional then $\dim W_\chi(H^\infty) \leq 1$.

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- The analogous result is true for quaternionic discrete series and the Heisenberg parabolic subgroup.
- These results follow from a general result of mine that proves a variant of multiplicity one in the sense above for representations induced from finite dimensional representations of what Karin Baur and I call "(very) nice parabolic subgroups".

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- For example $G = Sp(n, \mathbb{R})$ (rank n symplectic group), P the Siegel parabolic. N is isomorphic with the symmetric $n \times n$ matrices over $\mathbb{R}$. M is isomorphic with $GL(n, \mathbb{R})$ acting on N by $gX = gXg^T$. The unitary characters of N are of the form $\chi_S(X) = e^{itr(SX)}$ with S a symmetric matrix. The stabilizer of χ_S in M is $\{g \in GL(n, \mathbb{R}) | gSg^T = S\}$. χ_S is generic if $\det(S) \neq 0$. The stabilizer of χ_S is compact if and only if S is positive or negative definite.

- Raul Gomez and I have generalized these results to the case for general representations in the case when N is abelian and the stabilizer of χ is compact.
- For example $G = Sp(n, \mathbb{R})$ (rank n symplectic group), P the Siegel parabolic. N is isomorphic with the symmetric $n \times n$ matrices over $\mathbb{R}$. M is isomorphic with $GL(n, \mathbb{R})$ acting on N by $gX = gXg^T$. The unitary characters of N are of the form $\chi_S(X) = e^{itr(SX)}$ with S a symmetric matrix. The stabilizer of χ_S in M is $\{g \in GL(n, \mathbb{R}) | gSg^T = S\}$. χ_S is generic if $\det(S) \neq 0$. The stabilizer of χ_S is compact if and only if S is positive or negative definite.
- Important work from a similar perspective involving dual pairs has recently been done by Gomez and Wee Teck Gan and Gomez and Chen-bo Zhu including results in the non-archimedean case.

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- Roe Goodman and I showed that if $\lambda \in W_\chi(V)$ then it extends to a completion of V intermediate to Schmid's minimal completion (analytic vectors) and the C^∞–vectors. We also construct for each element of the Weyl group an element of $W_\chi(V)$ yielding a basis and show that the element corresponding to the longest element of the Weyl group has the same growth as a conical vector in the negative Weyl chamber (unfortunately not in the negative dual Weyl chamber).

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- Roberto Miatello and I proved the same result for groups of real rank one. That is if $I_{P,\xi,\nu}$ is a principal series representation and χ is a character of N there is a χ–Whittaker vector, $\lambda(\nu)$, on the analytic vectors with the property that it has the same growth properties as a conical vector. After applying an appropriate fudge factor $\lambda(\nu)$ is holomorphic in ν and non-zero for all ν.

- This implies that if $\Gamma \subset G$ were such that $N \cap \Gamma$ is co-compact in N and if χ is generic and trivial on $N \cap \Gamma$ then

$$\sum_{\gamma \in N \cap \Gamma \backslash \Gamma} \lambda(\nu)(\pi_\nu(\gamma g)f) = M_\nu(f)(g)$$

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- We prove a meromorphic continuation of $M_\nu(f)$. These functions satisfy the definition of an automorphic form except for the condition of moderate growth. However at the poles (usually simple) the residues satisfy the moderate growth condition and are elements of the discrete spectrum. Ideas similar to these for $SL(2, \mathbb{R})$ can be found in work of Good, Bruggeman and others.

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- I bring this up since the recent interest in the Ramanujan mock theta functions has led to an understanding that they are related to "automorphic forms" that are immoderate. Generalizations of these functions are being actively studied. I suggest that this work of mine an Miatello is probably related.


Happy Birthday Wilfried!