

Understanding restriction to K

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Outline

Understanding
restriction to K

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Introduction

Introduction

Discrete series

Discrete series

Standard representations

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Standard reps| K

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Why restrict to K ?

G cplx $\supset G(\mathbb{R})$ real reductive $\supset K(\mathbb{R})$ maxl compact

Reps (π, \mathcal{H}_π) of $G(\mathbb{R})$ are complicated and difficult.

Reps of $K(\mathbb{R})$ are easy, so try two things:

understand $\pi|_{K(\mathbb{R})}$; and

use understanding to answer questions about π .

$K(\mathbb{R}) \subset G(\mathbb{R}) \rightsquigarrow T \subset U$, max torus in cpt Lie.

$\mu \in \widehat{U}$ **characterized** by largest $\xi(\mu)$ in $\mu|_T$ (Cartan-Weyl).

Can **compute** $\mu|_T$ completely (Kostant).

When $\pi|_{K(\mathbb{R})}$ inf diml, can't ask for *largest* piece. . .

. . . can define $\mu(\pi)$ as a **smallest** piece of $\pi|_{K(\mathbb{R})}$.

(sometimes) π **characterized** by $\mu(\pi)$ (Schmid)

In those cases, can **compute** $\pi|_{K(\mathbb{R})}$ (Schmid)

Outline of talk

$G(\mathbb{R})$ real reductive $\supset K(\mathbb{R})$ max compact subgp.

Ex: $G(\mathbb{R}) = GL(n, \mathbb{C}) = \{\text{invertible linear transf of } \mathbb{C}^n\}$

$K(\mathbb{R}) = U(n) = \{\text{linear transf respecting } \langle, \rangle\}.$

Ex: $G(\mathbb{R}) = GL(n, \mathbb{R}) = \{\text{invertible linear transf of } \mathbb{R}^n\}$

$K(\mathbb{R}) = O(n) = \{\text{linear transf respecting } \langle, \rangle\}.$

Ex: $G(\mathbb{R}) = Sp(2n, \mathbb{R}) = \{\mathbb{R}\text{-lin transf of } \mathbb{C}^n \text{ resp. } \text{Im}(\langle, \rangle)\}$

$K(\mathbb{R}) = U(n) = \{\mathbb{C}\text{-linear maps in } Sp(2n, \mathbb{R})\}.$

Plan for this talk:

1. Recall **standard reps** $I(\gamma)$ of $G(\mathbb{R})$ (**Harish-Chandra**)
2. Recall **multiplicity calculation** of $I(\gamma)|_{K(\mathbb{R})}$ (**Schmid**)
3. Recall **(irred) = int comb (std reps)** (**Kazhdan-Lusztig**)
4. **geometric expression** for irr reps restricted to $K(\mathbb{R})$
5. **Open problems** relating 1)–3) to 4).

Example from Problem Session

$$G(\mathbb{R}) = SL(2, \mathbb{C}), K(\mathbb{R}) = SU(2).$$

$\tau(m)$ = irr rep of $K(\mathbb{R})$ of highest weight m .

$$\begin{aligned} I(m) &= \text{princ series rep } \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}(re^{i\theta} \mapsto e^{im\theta}) \\ &= \tau(m) + \tau(m+2) + \tau(m+4) \cdots \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{\text{reg}}(\text{triv}) &= \text{regular functions on } \mathcal{N}_{\theta}^* \\ &= \tau(0) + \tau(2) + \tau(4) + \cdots \\ &= I(0). \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{\text{reg}}(\text{nontriv}) &= \text{odd functions on double cover of } \mathcal{N}_{\theta}^* \\ &= \tau(1) + \tau(3) + \tau(5) + \cdots \\ &= I(1). \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{\text{zero}}(m) &= \text{copy of } \tau(m) \text{ at } 0 \in \mathcal{N}_{\theta}^* \\ &= \tau(m) \\ &= I(m) - I(m+2). \end{aligned}$$

Lusztig Conjecture bijection picks **largest** term on left. . .

$$I(0) \leftrightarrow (\text{reg, triv}), I(1) \leftrightarrow (\text{reg, nontriv}), I(m) \leftrightarrow (\text{zero}, m-2).$$

Setting

G cplx conn reductive alg grp $\overset{\text{cplx conj action } \sigma}{\text{def over } \mathbb{R}} \rightsquigarrow G(\mathbb{R})$
 σ_0 cpt form s.t. $\sigma\sigma_0 = \sigma_0\sigma \rightsquigarrow \theta =_{\text{def}} \sigma\sigma_0$ Cartan inv
 $K = G^\theta$ cplx reductive alg, $K(\mathbb{R})$ max cpt in $G(\mathbb{R})$.

$\Pi_u(G(\mathbb{R})) =$ irr unitary reps/equiv: atoms of harm analysis

\cap

$\Pi(G(\mathbb{R})) =$ irr quasisimple reps/infl equiv: analytic cont

\supset

$\Pi(\mathfrak{g}, K) =$ irr HC modules: Taylor series for qsimple reps

$\Pi(K(\mathbb{R})) = \Pi(K) = \widehat{K} =$ irr reps of $K(\mathbb{R}) =$ irr alg of K

$\pi \in \Pi(G(\mathbb{R})) \rightsquigarrow m_\pi: \widehat{K} \rightarrow \mathbb{N}$, $m_\pi(\mu) =$ mult of μ in $\pi|_K$.

Problem: compute and understand functions m_π .

Schmid's construction of discrete series

Discrete series reps $\Pi_{ds}(G(\mathbb{R})) \subset \Pi_u(G(\mathbb{R}))$

Discrete series are **irr summands of $L^2(G(\mathbb{R}))$** .

Harish-Chandra: exist iff $G(\mathbb{R}) \supset T(\mathbb{R}) \subset K(\mathbb{R})$, **cpt Cartan**

Harish-Chandra: $\widehat{T(\mathbb{R})}_{reg}/W(\mathbb{R}) \xrightarrow{\approx} \Pi_{ds}(G(\mathbb{R}))$, $\lambda \rightarrow I(\lambda)$

\mathcal{B} = **complete flag variety** of Borel subalgs $\mathfrak{b} \subset \mathfrak{g}$.

$\lambda \rightsquigarrow \mathfrak{b}(\lambda) \supset \mathfrak{t} \rightsquigarrow X(\lambda) = G(\mathbb{R}) \cdot \mathfrak{b}(\lambda) \simeq G(\mathbb{R})/T(\mathbb{R}) \subset \mathcal{B}$

∪

$Z(\lambda) = K(\mathbb{R}) \cdot \mathfrak{b}(\lambda) \simeq K(\mathbb{R})/T(\mathbb{R})$

$\mathcal{L}(\lambda + \rho) \rightarrow X(\lambda)$ holomorphic line bundle induced by $\lambda + \rho$.

Kostant-Langlands: $I(\lambda) \stackrel{?}{\approx} H^s(X(\lambda), \mathcal{L}(\lambda + \rho))$ ($s = \dim Z(\lambda)$).

Probs: $X(\lambda)$ noncpt; cohom not Hilbert space.

Schmid: **Taylor exp** of cohom along cpt subvar $Z(\lambda)$.

$I(\lambda)|_{K(\mathbb{R})} \approx H^s(Z(\lambda), \mathcal{L}(\lambda + \rho) \otimes \underbrace{S(\mathfrak{g}/(\mathfrak{k} + \mathfrak{b}(\lambda)))^*}_{\text{conorm to } Z \text{ in } X})$.

conorm to Z in X

Understanding Blattner

$G(\mathbb{R}) \supset K(\mathbb{R}) \supset T(\mathbb{R})$, \mathcal{B} = variety of Borel subalgs.

$\lambda \in \widehat{T(\mathbb{R})}_{\text{reg}} \rightsquigarrow \mathfrak{b}^-(\lambda) \supset \mathfrak{t}$ Borel subalgebra.

$X(\lambda) = G(\mathbb{R}) \cdot \mathfrak{b}^-(\lambda) \simeq G(\mathbb{R})/T(\mathbb{R})$ open $G(\mathbb{R})$ orbit.

\rightsquigarrow finitely many $G(\mathbb{R})$ orbits on $\mathcal{B} \rightsquigarrow$ reps by geom quant.

$$I(\lambda)|_{K(\mathbb{R})} \approx H^s(Z(\lambda), \mathcal{L}(\lambda + \rho) \otimes \mathcal{S}(\mathfrak{g}/(\mathfrak{k} + \mathfrak{b}(\lambda))))^*.$$

Starts with lowest K -type $H^s(K/(B \cap K), \mathcal{L}(\lambda + \rho))$.

This is irr of highest weight $\mu(\lambda) = \lambda + \rho - \underbrace{2\rho_c}_{\text{sum cpt pos}}$

Borel-Weil-Bott-Kostant \rightsquigarrow entire restriction $I(\lambda)|_{K(\mathbb{R})}$.

Thm (Schmid, Hecht-Schmid) Discrete series $I(\lambda)$ contains lowest K -type $\mu(\lambda)$ with mult 1; all others are $\mu(\lambda) + (\text{sum ncpt pos})$. These props characterize $I(\lambda)$.

Standard representations

$G(\mathbb{R}) \supset H(\mathbb{R})$, **Cartan subgp.** After conj \rightsquigarrow θ -stable.

$H(\mathbb{R}) = \underbrace{T(\mathbb{R})}_{\text{cpt}} \times \underbrace{A}_{\text{vec gp}}$, **cplxification** $T = H^\theta = H \cap K$.

$\widehat{T(\mathbb{R})} = X^*(T) = \underbrace{X^*(H)/(1-\theta)X^*(H)}_{\text{small if } H \text{ ncpt}}$, $\widehat{A} = \underbrace{\mathfrak{a}^*}_{\text{big if } H \text{ ncpt}}$.

Harish-Chandra: $\widehat{H(\mathbb{R})}/W(\mathbb{R}) \rightsquigarrow \Pi_{\text{std}}(G(\mathbb{R}))$, $\gamma \rightarrow I(\gamma)$

Need also $\Psi =$ **pos imag roots** making λ **dom.**

$\gamma = (\lambda, \nu) \in \widehat{T(\mathbb{R})} \times \widehat{A}$

1. $I(\gamma, \Psi)$ **tempered** $\iff \gamma$ unitary $\iff \nu \in i\mathfrak{a}_0^*$
2. $I(\gamma, \Psi)|_K$ depends only on $\lambda \in X^*(T)$

$I(\gamma, \Psi)|_K =_{\text{def}} I(\lambda, \Psi)$ **known:** hard case **disc series.**

Basis for virtual reps

$H(\mathbb{R}) = T(\mathbb{R}) \times A$, $\gamma = (\lambda, \nu) \in \widehat{H(\mathbb{R})}$, Ψ pos imag.

All $I(\gamma, \Psi)$ lin ind **unless** $\langle \gamma, \alpha^\vee \rangle = 0$ ($\alpha \in \Psi$ simple)...

... **Hecht-Schmid character identities**

1. **α noncompact:**

$$\underbrace{I(\gamma_\alpha, \Psi_\alpha)}_{\text{spl } H_\alpha \subset SL(2)_\alpha} = \begin{cases} I(\gamma, \Psi) + I(\gamma, s_\alpha \Psi) & s_\alpha \notin W(\mathbb{R}) \\ I(\gamma, \Psi) & s_\alpha \in W(\mathbb{R}). \end{cases}$$

2. **α compact:** $I(\gamma, \Psi) = 0$.

(γ, Ψ) **final** if not on left of a Hecht-Schmid identity.

$\iff \nexists \alpha$ real, $\langle \nu, \alpha^\vee \rangle = 0$, $\lambda(m_\alpha) = -1$, **and**

$\nexists \alpha$ compact simple in Ψ , $\langle \lambda, \alpha^\vee \rangle = 0$.

Thm (Langlands, Knapp-Zuckerman, Hecht-Schmid)

1. $\{I(\gamma, \Psi) \mid (\gamma, \Psi) \text{ final}\}$ **basis** for virtual reps of $G(\mathbb{R})$.
2. $\{I(\gamma, \Psi) \mid (\gamma, \Psi) \text{ final, unitary}\} = \Pi_{temp}(G(\mathbb{R}))$.
3. $I(\gamma, \Psi)$ has quotient rep $J(\gamma, \Psi)$; **irr** if (γ, Ψ) final.

Irreducibles and standards

$\mathcal{P}(G(\mathbb{R})) = \{(\gamma, \Psi) \text{ final} \mid \gamma \in \widehat{H(\mathbb{R})}, \Psi \text{ pos imag}\}$

Langlands parameters; write $x \in \mathcal{P}$ for (γ, Ψ) .

$I(x)$ standard rep $\rightarrow J(x)$ irr **Langlands quotient**.

$\mathcal{P}(G(\mathbb{R}))$ **basis** / $\mathbb{Z}[q, q^{-1}]$ for Hecke alg module.

KL analysis \rightsquigarrow Two kinds of KL polys in $\mathbb{N}[q]$. . .

1. $Q_{z,y}(1) =$ **mult** of $J(z)$ in $I(y)$

$$I(y) = \sum_{z \leq y} Q_{z,y}(1) J(z).$$

2. $P_{y,x}(1) = (-1)^{\ell(x) - \ell(y)}$. (**coeff** of $I(y)$ in char of $J(x)$)

$$J(x) = \sum_{y \leq x} (-1)^{\ell(x) - \ell(y)} P_{y,x}(1) I(y).$$

(x, y, z in $\mathcal{P}(G(\mathbb{R}))$).

Consequence: computable branching law

$$J(x)|_K = \sum_{y \leq x} (-1)^{\ell(x) - \ell(y)} P_{y,x}(1) I(y)|_K.$$

Basis for restrictions to K

$\{I((\lambda, \nu), \Psi)\}$ are all the standard reps.

Restriction to K **independent** of continuous param ν .

Thm (How standard reps restrict to K .)

1. $\{I(\lambda, \Psi) \mid \lambda \in X^*(T) \text{ final}\}$ **basis** for (virtual reps) $|_K$.
2. $I(\lambda, \Psi)$ has **unique** lowest K -type $\mu(\lambda, \Psi)$.
3. $\{\text{temp, real infl char}\} \leftrightarrow \{I(\lambda, \Psi) \mid \lambda \in X^*(T) \text{ final}\} \leftrightarrow \widehat{K}$

Ex: $G = SL(2) \times SL(2)$, $K = SL(2)_\Delta$, torus = $H \times H$

$T = H_\Delta \rightsquigarrow$ final params (temp reps $SL(2, \mathbb{C})$, real infl)...

$\{n \mid n \geq 0\}$, $I(n)|_{SL(2)_\Delta} = \{E(n), E(n+2), E(n+4) \dots\}$

Ex: $G = SL(2)$, $K = SO(2)$, $H_c = K$, $H_s =$ (diag torus)

$T_c = K$, Ψ^{hol}, Ψ^{ahol} ; $T_s = \{\pm I\} \rightsquigarrow$ final params...

$\{(n, \Psi^{hol}) \mid n \geq 0\}$ on T_c ; $\{(m, \Psi^{ahol}) \mid m \leq 0\}$ on T_c ; (triv) on T_s .

$$I(n, \Psi^{hol})|_{SO(2)} = \{n+1, n+3, n+5 \dots\}$$

$$I(m, \Psi^{ahol})|_{SO(2)} = \{m-1, m-3, m-5 \dots\}$$

$$I(\text{triv})|_{SO(2)} = \{0, \pm 2, \pm 4 \dots\}$$

$\mathcal{F}(\mathfrak{g}, K)$ category of finite length (\mathfrak{g}, K) -modules:
 $U(\mathfrak{g})$ -module, alg action of $K = G^\theta$.

$\xrightarrow{\text{gr}} \mathcal{C}(\mathfrak{g}, K)$ f.g. $(\mathcal{S}(\mathfrak{g}/\mathfrak{k}), K)$ -mods, $\text{supp} \subset \mathcal{N}_\theta^* \subset (\mathfrak{g}/\mathfrak{k})^*$

$\mathcal{N}^* = \{\lambda \in \mathfrak{g}^* \mid \rho(\lambda) = 0 \text{ } (\rho \in [\mathfrak{g}\mathcal{S}(\mathfrak{g})]^G)\}$ **nilp cone**

$\mathcal{N}_\theta^* = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$, $\mathcal{N}_{\mathbb{R}}^* = \mathcal{N}^* \cap i\mathfrak{g}(\mathbb{R})^*$.

Prop (Kostant-Rallis, Sekiguchi)

1. K acts on \mathcal{N}_θ^* , **fin # orbs**, cplx Lag in G orbit.
2. $G(\mathbb{R})$ acts on $\mathcal{N}_{\mathbb{R}}^*$, **fin # orbs**, real Lag in G orbit.
3. **Bij** $\mathcal{N}_\theta^*/K \leftrightarrow \mathcal{N}_{\mathbb{R}}^*/G(\mathbb{R})$, resp G orbit, diffeo type.

Prop **gr** induces **surjection** of Grothendieck groups

$$K\mathcal{F}(\mathfrak{g}, K) \xrightarrow{\text{gr}} K\mathcal{C}(\mathfrak{g}, K);$$

image records restriction to K of HC module.

So **restrictions to K of HC modules** sit in interesting category: **coherent sheaves on nilp cone in $(\mathfrak{g}/\mathfrak{k})^*$.**

gr for discrete series

Recall constr of *disc ser reps* starts with maxl torus $T \subset K$ (cplx alg), reg char $\lambda \in X^*(T)/W(K, T)$.

\mathcal{B} = complete flag variety of Borel subalgs $\mathfrak{b} \subset \mathfrak{g}$.

$\lambda \rightsquigarrow \mathfrak{b}(\lambda) \supset \mathfrak{t} \rightsquigarrow Z(\lambda) = K \cdot \mathfrak{b}(\lambda) \simeq K/K \cap B(\lambda) \subset \mathcal{B}$

$\mathcal{L}(\lambda + \rho) \rightarrow \mathcal{B}$ algebraic line bundle induced by $\lambda + \rho$.

\mathcal{D} -module picture: $I(\lambda) =$ formal conormal derivatives of hol secs of $\mathcal{L}(\lambda + \rho)$ on closed K -orbit $Z(\lambda) \subset \mathcal{B}$

Recall Schmid: Taylor exp along cpt subvar $Z(\lambda)$:

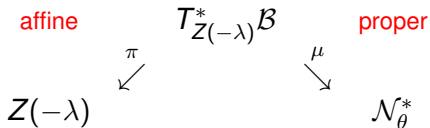
$$I(\lambda)|_K \approx H^s(Z(\lambda), \mathcal{L}(\lambda + \rho) \otimes \mathcal{S}(\underbrace{\mathfrak{g}/[\mathfrak{k} + \mathfrak{b}(\lambda)]}_{\text{conorm to } Z \text{ in } X})^*).$$

Serre duality, etc. \rightsquigarrow

$$\begin{aligned} \text{gr } I(\lambda) &\simeq H^0(Z(-\lambda), \mathcal{L}(\lambda + \rho - 2\rho_c) \otimes \mathcal{S}(\mathfrak{g}/[\mathfrak{k} + \mathfrak{b}(-\lambda)])) \\ &\simeq \text{pullback of } \mathcal{L}(\lambda + \rho - 2\rho_c) \text{ to } T_{Z(-\lambda)}^* \mathcal{B} \end{aligned}$$

Associated varieties for standard reps

Recall $\text{gr } I(\lambda) \simeq$ pullback of $\mathcal{L}(\lambda + \rho - 2\rho_c)$ to $T_{Z(-\lambda)}^* \mathcal{B}$.



$\text{gr } I(\lambda) \simeq \mu_* \pi^* \mathcal{L}(\lambda + \rho - 2\rho_c)$, coh sheaf on $\mu(T_{Z(-\lambda)}^*)$.

Similarly $\text{gr } I(\gamma, \Psi) \rightsquigarrow$ conorm geom $\rightsquigarrow K \setminus \mathcal{N}_\theta^*$.

Two bases for $K\mathcal{C}(\mathfrak{g}, K)$:

Langlands $[(\lambda, \Psi) \text{ final}, \lambda \text{ char of } H^\theta] / K \rightsquigarrow \text{gr } I(\lambda, \Psi)$
geometric orbit $K \cdot \xi \subset \mathcal{N}_\theta^*$, irr rep τ of $K^\xi \rightsquigarrow \Gamma[K \times_{K^\xi} E_\tau]$

Prob 1: *calculate* chg-bas-mtrx; KL \rightsquigarrow assoc var(irr).

Prob 2: *understand* chg-bas-mtrx; prove (nearly) triang, \rightsquigarrow (nearly) **bijection** between bases.

Problem 2 due to Lusztig in case $G(\mathbb{R})$ complex; resolved by Bezrukavnikov, Ostrik.

Being guided through representation theory



$$m_{\pi(\lambda)}(\mu) = \sum_{w \in W_K} \epsilon(w) P_n([w(\mu + \rho_c) - \rho_c] - [\lambda + \rho - 2\rho_c])$$

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