Endoscopy, inner twists and characters

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Reference:

The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups,

to appear in

Colloquium Publications, AMS.

I F number field, $GL(N)/F$

Consider the set $\Psi_{sim}(N) = \Psi_{sim}(GL(N))$ consisting of

(i) A decomp. $N = mn$.

(ii) $\mu \in \Pi_{cusp}(GL(m))$, unitary.

(iii) $\nu : SU(2) \longrightarrow GL(n, \mathbb{C})$ irred.

THEOREM (Moeglin-Waldspurger). There is a bijection

$$\psi \in \Psi_{sim}(N) \longleftrightarrow \pi_\psi \subset L^2_{disc}(GL(N,F)\backslash GL(N,\mathbb{A}))$$

COROLLARY: (Langlands, Eisenstein series)

Let $\Psi(N)$ be the set consisting of

(i) A partition $N = N_1 + \cdots + N_r$.

(ii) A formal unordered sum $\psi = \psi_1 \boxplus \cdots \boxplus \psi_r$, $\psi_i \in \Psi_{sim}(N_i)$.

Then there is a bijection

$$\psi \in \Psi(N) \longleftrightarrow \pi_\psi \subset L^2(GL(N,F)\backslash GL(N,\mathbb{A}))$$

$$\psi \longleftrightarrow \pi_\psi \longrightarrow c(\psi) = c(\pi_\psi) = \big(c_v(\psi) = (c(\pi_{\psi,v})) : v \notin S\big)$$

$\sim$ equivalence classes of s.s. conj. classes $c = (c_v)$ in $GL(N,\mathbb{C})$, where $c \sim c'$ if $c_v = c'_v$ for a. a. v.

<u>THEOREM</u>: (Jacquet - Shalika) Set $\mathcal{C}(N) = \{c(\psi) : \psi \in \Psi(N)\}$. Then $\psi \longrightarrow c(\psi)$ is a bijection

<u>Remarks</u> (i) This takes the subset

$$\widehat{\Psi}(N) = \{\psi \in \Psi(N) : \pi_\psi \cong \pi_\psi^\vee \text{ (contragredient)}\}$$

onto

$$\widehat{\mathcal{C}}(N) = \{c \in \mathcal{C}(N) : c_v = c_v^{-1},\ v \notin S\}.$$

(ii) We expect that

$\widehat{\Psi}(N) \longleftrightarrow \{\psi : L_F \times SU(2) \longrightarrow GL(N,\mathbb{C})\} \sim$ unitary, self-dual, N-dim. rep^{s}, where L_F is the hypothetical automorphic Langlands group.

(II) G/F, conn. quasisplit, orthog or symplectic group (F- global)

B_n: $G = SO(2n+1)$ - split; $\hat{G} = Sp(2n, \mathbb{C}) = {}^LG \subset GL(N, \mathbb{C})$, $N = 2n$.

C_n: $G = Sp(2n)$ - split; $\hat{G} = SO(2n+1, \mathbb{C}) = {}^LG \subset GL(N, \mathbb{C})$, $N = 2n+1$.

D_n: $G = SO(2n)$ - quasisplit; $\hat{G} = SO(2n, \mathbb{C})$,

${}^LG = \hat{G} \rtimes Gal(E/F) \cong O(2n, \mathbb{C}) \subset GL(N, \mathbb{C})$, $N = 2n$,

$deg(E/F) = 1, 2$.

Set

$\tilde{\mathcal{C}}(G) = \{ c(\pi) = (c_v(\pi) = c(\pi_v)) : \pi \text{ (irred)} \subset L^2(G(F) \backslash G(\mathbb{A})) \}$

- families of s.s. classes in ${}^LG$, taken up to conjugacy by $\hat{G}$ in case B_n + C_n, and by $O(2n, \mathbb{C})$ (rather than by $\hat{G} = SO(2n, \mathbb{C})$) in case D_n (and with the eq. rel^n $c \sim c'$ above)

THEOREM: (Global endoscopy 1). (i) The embedding ${}^LG \subset GL(N, \mathbb{C})$ gives a mapping

$$\widetilde{\mathcal{C}}(G) \longrightarrow \widehat{\mathcal{C}}(N)$$

(ii) The mapping is injective

(iii) There is a simple description of its image – i.e. of the subset

$$\widehat{\Psi}(G) = \{\psi \in \widehat{\Psi}(N) : c(\psi) \in \widetilde{\mathcal{C}}(G) \subset \widehat{\mathcal{C}}(N)\} \quad \text{of } \widehat{\Psi}(N).$$

Part (a) of global (twisted) endoscopy for $GL(N)$.

Caution: The mapping $\widetilde{\pi}$ (vired) $\subset L^2(G(F)\backslash G(\mathbb{A})) \longrightarrow c(\widetilde{\pi}) \in \widetilde{\mathcal{C}}(G) \cong \widetilde{\Psi}(G) \subset \widehat{\Psi}(N)$

is not injective

- Remaining parts

(b) For any $\psi \in \widehat{\Psi}(G)$, describe its preimage $\widetilde{\Pi}_\psi \subset \widetilde{\Pi}_{unit}(G)$

(c) For any $\widetilde{\pi} \in \widetilde{\Pi}_\psi$, describe its multiplicity in $L^2_{disc}(G(F)\backslash G(\mathbb{A}))$.

III) G/F as above, but F local

$$L_F = \begin{cases} W_F, & F \text{ archimedean} \\ W_F \times SU(2), & F \text{ p-adic} \end{cases} \quad : \text{ local Langlands group}$$

- $\Phi(G) = \{ \phi : L_F \longrightarrow {}^L G$, L-homo$^{\text{ism}}$ up to $\hat{G}$-conjugacy $\}$
- $\Pi(G) = \{$ irred. rep^s π of $G(F)$, up to equivalence $\}$

- $\tilde{\Phi}(G) = \Phi(G)/\sim$ $\quad$ • $\tilde{\Pi}(G) = \Pi(G)/\sim$,

  for equiv. rel^n $\sim$ trivial in cases B_n + C_n, but defined by conjugacy by $O(2n)$ instead of $SO(2n)$ in case D_n.

- $\tilde{\Phi}_{bdd}(G) = \{ \phi \in \tilde{\Phi}(G) : \phi(L_F)$ is rel. compact $\}$
- $\tilde{\Pi}_{temp}(G) = \{ \pi \in \tilde{\Pi}(G) : \pi$ tempered $\}$

- $\tilde{\Psi}(G) = \{ \psi : L_F \times SU(2) \longrightarrow {}^L G$, with $\psi|_{L_F} \in \tilde{\Phi}_{bdd}(G) \}$
- $\tilde{\Pi}_{unit}(G) = \{ \pi \in \tilde{\Pi}(G) : \pi$ unitary $\}$

- Similar definitions for $GL(N)$: $\Phi(N) = \Phi(GL(N))$,

$$\Phi_{bdd}(N) = \Phi_{bdd}(GL(N)), \quad \Psi(N) = \Psi(GL(N)).$$

<u>THEOREM</u>: (Harris-Taylor, Henniart, Scholze) There is a unique bijection $\phi \longrightarrow \pi_\phi$ from $\Phi_{bdd}(N)$ onto $\Pi_{temp}(N)$ that is compatible with local Rankin-Selberg L-functions and ε-factors, and also with the standard automorphism

$$\widehat{\theta}(N) : g \longrightarrow \widetilde{J}(N)\, {}^t g^{-1}\, \widetilde{J}(N)^{-1}, \qquad g \in GL(N).$$

i.e. it is a bijection between the self-dual subsets

$$\widetilde{\Phi}_{bdd}(N) \subset \Phi_{bdd}(N) \xrightarrow{\sim} \widetilde{\Pi}_{temp}(N) \subset \Pi_{temp}(N)$$

If $\psi \in \widetilde{\Psi}(G)$, define

$S_\psi = \mathrm{Cent}(\mathrm{im}(\psi), \hat{G})$ ~ reductive gp. $/\mathbb{C}$

$\mathcal{S}_\psi = S_\psi / S_\psi^0 Z(\hat{G})^\Gamma$ – finite abelian 2-group.

<u>THEOREM</u> (Local endoscopy 1). (i) For any $\psi \in \widetilde{\Psi}(G)$, there is a finite "multi-set" $\widetilde{\Pi}_\psi$ in $\widetilde{\Pi}_{unit}(G)$ (i.e. a finite set over $\widetilde{\Pi}_{unit}(G)$), with a canonical mapping

$$\pi \in \widetilde{\Pi}_\psi \longrightarrow \langle \cdot, \pi \rangle \in \hat{\mathcal{S}}_\psi,$$

both determined by twisted character rel^{n} from $GL(N)$.

(ii) Suppose that $\phi = \psi$ lies in the subset

$$\widetilde{\Phi}_{bdd}(G) = \{\psi : \psi|_{SU(2)} = 1\}$$

of $\widetilde{\Psi}(G)$. Then the el^{ts} in $\widetilde{\Pi}_\phi$ are <u>tempered</u>, <u>mult. free</u>, and the mapping $\widetilde{\Pi}_\phi \longrightarrow \hat{\mathcal{S}}_\phi$ is <u>injective</u>, and <u>bijective</u> if F is p-adic.

Moreover,
$$\widetilde{\Pi}_{temp}(G) = \coprod_{\phi \in \widetilde{\Phi}_{bdd}(G)} \widetilde{\Pi}_\phi.$$

IV Local packets and characters

For simplicity, take $\phi = \psi$ in $\tilde{\Phi}_{bdd}(G) \subset \tilde{\Psi}(G)$.

- $\mathrm{tr}(\pi(f)) = \int_{G_{reg}(F)} \Theta_G(\pi, x) f(x)\, dx$, $\quad f \in \mathcal{H}(G)$, $\pi \in \Pi(G)$.

$\Theta_G(\pi, \cdot)$ character of π, analytic on $G_{reg}(F)$, loc. integ. on $G(F)$.

- $I_G(\pi, \gamma) \doteq |D(\gamma)|^{\frac{1}{2}} \Theta_G(\pi, \gamma)$, $\quad \gamma \in \Gamma_{reg}(G)$ – (reg. conj. classes in $G(F)$)

normalized character

Variants

(i) $\tilde{I}_G(\pi, \gamma) \doteq \sum_{\pi_* \to \pi} I_G(\pi_*, \gamma)$, $\quad \pi \in \tilde{\Pi}(G)$, $\gamma \in \tilde{\Gamma}_{reg}(G)$ (orbits in $\Gamma_{reg}(G)$, under $\tilde{\theta} \in \mathrm{Out}(G)$)

(ii) $\tilde{I}_N(\tilde{\pi}_\phi, \tilde{\gamma})$ – twisted character on $\tilde{G}(N, F) = GL(N, F) \rtimes \tilde{\theta}(N)$, $\tilde{\gamma} \in \tilde{\Gamma}_{reg}(N)$ – $GL(N, F)$-orbits in $\tilde{G}(N, F)$.

(iii) $\tilde{S}^G(\phi, \delta) \doteq \sum_{\tilde{\gamma} \in \tilde{\Gamma}_{reg}(N)} \tilde{I}_N(\tilde{\pi}_\phi, \tilde{\gamma}) \overline{\Delta(\delta, \tilde{\gamma})}$,

stable character of ϕ on $G(F)$, $\delta \in \tilde{\Delta}_{reg}(G)$ ($\tilde{\theta}$-orbits of stable reg. conj. classes in $G_{reg}(F)$), $\Delta(\delta, \tilde{\gamma})$ – Kottwitz-Shelstad twisted transfer factor

<u>General property of endoscopy for G</u>: there is bijective corresp.

$$(G', \phi') \longleftrightarrow (\phi, s), \qquad \phi \in \Phi_{bdd}(G),\ s \in S_{\phi, ss},$$

where G' is an endoscopic group for G, + $\phi' \in \Phi_{bdd}(G')$.

Since $\hat{G}' = \mathrm{Cent}(s, \hat{G})^0$ is a product of complex groups GL and SO or Sp, the stable character

$$\hat{S}^{G'}(\phi', \delta'), \qquad \delta' \in \hat{\Delta}_{reg}(G'),$$

on $G'(F)$ is defined as above.

<u>THEOREM</u>: (Local endoscopy 2) Suppose $\phi \in \hat{\Phi}_{bdd}(G)$ + $\pi \in \widehat{\Pi}_\phi$. Then

$$\tilde{\Phi}_G(\pi, \gamma) = \sum_{x \in \mathcal{S}_\phi} \sum_{\delta' \in \tilde{\Delta}_{reg}(G')} \langle x, \pi \rangle^{-1} \hat{S}^{G'}(\phi', \delta') \, \Delta(\delta', \gamma),$$

where $(G', \phi') \longleftrightarrow (\phi, s)$, for any $s \in S_{\phi, ss}$ that maps to $x \in \mathcal{S}_\phi$, + $\Delta(\delta', \gamma)$ is the Langlands-Shelstad transfer factor for (G, G')

Remarks:

- The general case of $\psi \in \widehat{\Psi}(G)$ & $\pi \in \widehat{\Pi}_\psi$ is similar.

- Similar results apply to parameters ψ in the larger set $\widehat{\Psi}^+(G)$ (defined without cond^n that $\psi|_{L_F} \in \widehat{\Phi}_{bdd}(G)$), except the elements in $\widehat{\Pi}_\psi$ could be reducible and nonunitary. This is needed for global results, to account for possible failure of Ramanujan for $GL(N)$.

(V) G/F as above, F global (again), $v \in \mathrm{val}\, F$

$$\psi \in \Psi(N) \underset{(M\text{-}W,\, L)}{\rightsquigarrow} \pi_\psi \in \Pi(N) \underset{(\text{localize})}{\longrightarrow} \pi_{\psi,v} \in \Pi_v(N) \underset{(H\text{-}T,\, H,\, S)}{\rightsquigarrow} \psi_v \in \Psi_v^+(N).$$

<u>PROP</u>: If $\psi \in \widetilde{\Psi}_{sim}(G) \subset \Psi_{sim}(N)$, then ψ_v lies in $\widetilde{\Psi}^+(G_v) \subset \Psi_v^+(N)$.

i.e. ψ_v maps $L_{F_v} \times SU(2)$ to the subgroup ${}^L G_v$ of $GL(N, \mathbb{C})$.

Let $\widetilde{\Psi}_2(G)$ be the subset of global elements

$$\psi = \psi_1 \boxplus \cdots \boxplus \psi_r, \qquad \psi_i \in \widetilde{\Psi}_{sim}(G_i) \text{ } \underline{\text{self-dual}} \text{ \& } \underline{\text{distinct}},$$

in $\widetilde{\Psi}(G) \subset \widetilde{\Psi}(N)$. If $\psi \in \widetilde{\Psi}_2(G)$, one can define an extension

$$1 \longrightarrow \hat{G}_1 \times \cdots \times \hat{G}_r \longrightarrow \mathcal{G}_\psi \longrightarrow \Gamma_{E_\psi/F} \longrightarrow 1, \qquad E_\psi = E_1 \cdots E_r,$$

with an L-embedding

$$\mathcal{G}_\psi \subset \hat{G} \rtimes \Gamma_{E/F} = {}^L G \subset GL(N, \mathbb{C}).$$

This in turn yields finite abelian 2-groups

$$S_\psi = \mathrm{Cent}(\mathcal{G}_\psi, \hat{G}) \quad \underline{\text{and}} \quad \mathcal{S}_\psi = S_\psi / Z(\hat{G})^\Gamma.$$

- If $v \in \mathrm{val}(F)$, ψ_v maps $L_{F_v} \times SU(2)$ to the sub gp $\mathcal{G}_\psi$ of ${}^L G$,

  so that $\psi_v \in \widetilde{\Psi}^+(G_v) \subset \widetilde{\Psi}^+_v(N)$ (by the proposition).

- We thus get mappings

  $$x \in \mathcal{S}_\psi \longrightarrow x_v \in \mathcal{S}_{\psi_v}, \qquad v \in \mathrm{val}(F)$$

- We also get a global packet

  $$\widetilde{\Pi}_\psi = \{\pi = \bigotimes_v \pi_v : \pi_v \in \widetilde{\Pi}_{\psi_v},\ \langle \cdot, \pi_v \rangle = 1, \text{ for a.all } v\}.$$

- Any $\pi = \bigotimes_v \pi_v$ in $\widetilde{\Pi}_\psi$ then has a character on $\mathcal{S}_\psi$ -

  $$\langle x, \pi \rangle = \prod_v \langle x_v, \pi_v \rangle, \qquad x \in \mathcal{S}_\psi.$$

- $\widetilde{\mathcal{H}}(G) = \widetilde{\bigotimes_v} \mathcal{H}(G_v)$ – locally symmetric Hecke alg. on $G(\mathbb{A})$, relative to the outer aut$^{\underline{\text{m}}}$ $\hat{\theta}$ of $SO(2n)$ in case $\underline{\underline{D_n}}$.

<u>THEOREM</u>: (Global endoscopy 2) There is an $\widetilde{\mathcal{H}}(G)$-module iso$^{\underline{\text{m}}}$

$$L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})) \cong \bigoplus_{\psi \in \widetilde{\Psi}_2(G)} \bigoplus_{\{\pi \in \Pi_\psi : \langle \cdot, \pi \rangle = \varepsilon_\psi\}} m_\psi \pi,$$

where

$$m_\psi \in \{1, 2\},$$

and

$$\varepsilon_\psi : \mathcal{S}_\psi \longrightarrow \{\pm 1\}$$

is a (linear) character defined explicitly in terms of symplectic root numbers.

<u>Definitions of m_ψ + ε_ψ</u>

- $m_\psi = |\Phi(G,\psi)| = \{\psi_G \in \Phi(G) : \psi_G \longrightarrow \psi\}$

$$= \begin{cases} 2, & \text{if } G = SO(2n),\ N_i = \deg(\psi_i) \text{ is } \underline{\text{even}}\ \forall \psi_i \\ 1, & \text{otherwise.} \end{cases}$$

- Define

$$\tau_\psi : S_\psi \times L_F \times SU(2) \longrightarrow GL(\hat{\mathfrak{g}}), \qquad \hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G}),$$

by $\tau_\psi(s, g, h) = \mathrm{Ad}(s\,\psi(g \times h))$.

If

$$\tau_\psi = \bigoplus_\alpha (\lambda_\alpha \otimes \mu_\alpha \otimes \nu_\alpha) \quad \text{- irred. decomp, then}$$

$$\varepsilon_\psi(s) = \prod_\alpha{}' \det(\lambda_\alpha(s)),$$

where $\prod'$ is the product over indices α with

μ_α <u>symplectic</u> and $\varepsilon(\tfrac{1}{2}, \mu_\alpha) = -1$.